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FINITE ELEMENT APPROXIMATION OF LARGE AIR POLLUTION PROBLEMS I: ADVECTION

by

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April 1995

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FINITE ELEMENT APPROXIMATION OF LARGE AIR POLLUTION PROBLEMS I: ADVECTION

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Abstract

An Eulerian and semi-Lagrangian finite element methods for the solution of the two dimensional advection equation were developed. Bilinear rectangular elements were used. Linear stability analysis of the method is given.

1. Introduction

Two photographs appearing in the New York Times (March 28, 1994) show the damage of air pollution near big emission sources. But the problem exists even away from sources since air pollutants can be transported, mainly by advection. Thus air pollution becomes a global problem. This physical phenomenon consists of three major stages (see e.g. Zlatev [1]):

1. emission,
2. transport/advection,
3. transformation during the transport which includes: diffusion, deposition and chemical reactions.

In this paper, we only discuss the transport stage and the solution of the two dimensional advection equation by finite element methods.

2. Finite Element Solution

The two dimensional advection equation is given by

$$\frac{\partial c}{\partial t} = -\frac{\partial}{\partial x}(uc) - \frac{\partial}{\partial y}(vc), \qquad x_L \le x \le x_R, \quad y_L \le y \le y_U, \quad 0 < t \le T \quad (1)$$

where c is the concentration of a certain pollutant and u and v are the wind velocity components in the x and y directions, respectively. Clearly, when one is interested in several pollutants, the equation is replaced by a system of such equations coupled only via the chemical interaction between species.

The methods for numerical solution of the advection equation can be divided into five groups:

1. Finite differences,
2. Spectral methods,
3. Finite volume,
4. Characteristic-based methods or semi-Lagrangian,
5. Finite elements.

The finite difference methods are most popular and have been analyzed thoroughly (see e.g Richtmeyer and Morton [2]). Spectral methods (see e.g Orszag [3,4]) are used in weather forecasting, but not very much in air pollution. The pseudo spectral methods are of the same group. Here the solution is approximated by a truncated polynomial whose derivatives are substituted in the equation. The spectral methods require periodic boundary conditions. Finite volume or cell method is based on the integral form of the equation. The computational domain is divided into elements (volumes or cells) within which the integration is carried out. This method preserves the property of conservation (see Peyret and Taylor [11]). The semi-Lagrangian methods are not very popular among scientists working with air pollution models, but these are now gaining popularity in weather prediction. "Discretization schemes based on a semi-Lagrangian treatment of advection have elicited considerable interest... since they offer the promise of allowing larger time steps (with no loss of accuracy) than Eulerian-based schemes whose time step length is overly limited by consideration of stability" (see Staniforth and Côté [9]). Semi-Lagrangian methods based on finite difference or finite element spatial discretization were developed.

Here we discuss the finite element approximation to the two dimensional advection equation. Both Eulerian and semi-Lagrangian finite elements will be discussed and tested. Software will be available upon request or electronically via world wide web at the URL address http://math.nps.navy.mil/~bneta. The advantage of finite elements is the fact that the discretization can be as easily carried out for nonuniform grids. Thus one can use a fine grid only where the action is and a coarser grid away from there. First order linear one dimensional elements have been previously used, see e.g Pepper et al [5]. We now discuss bilinear finite elements on rectangles. It was shown by Neta and Williams [6] that isosceles triangles with linear basis functions and rectangular bilinear elements are superior to other triangulations and to finite differences. If the grid is uniform, rectangular elements are preferred since Staniforth et al [7] have shown how to evaluate the integrals efficiently and the mass matrix can be replaced by a tensor product of two tridiagonal matrices. If the grid is nonuniform again the rectangular elements are preferred, since the isosceles triangles lose their shape.

3. Bilinear Finite Elements

Discretize the rectangular domain, by introducing the nodes

$$(x_i, y_j), \quad i = 0, 1, \ldots, I+1, \quad j = 0, 1, \ldots, J+1,$$

where

$$x_0 = x_L,\ x_{I+1} = x_R,\ y_0 = y_L,\ y_{J+1} = y_U. \tag{2}$$

Suppose we number the interior nodes

$$n = 1, \ldots, IJ \tag{3}$$

from bottom left to top right, see figure 3 for the case $I = 4, J = 3$.

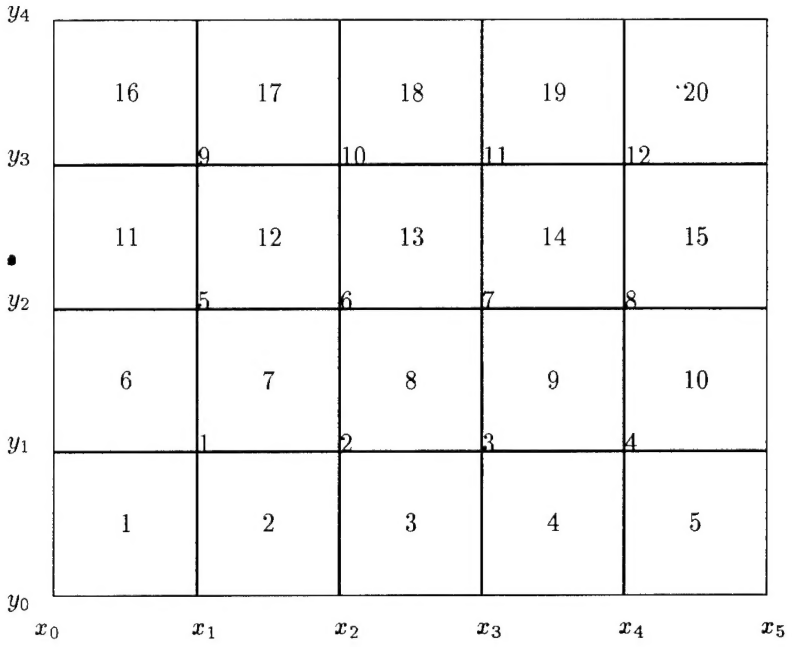

Figure 3: node and element numbering

The number of finite (rectangular) elements is $N_e = (I+1)(J+1) = 20$ in this case.

We now define the basis functions $\varphi_m(x, y)$ as bilinear functions on each rectangle, so that

$$\varphi_m(x, y) = \begin{cases} 1 & \text{at node } m \\ 0 & \text{at all other nodes.} \end{cases} \tag{4}$$

To obtain the bilinear basis functions defined on the k^{th} element, we can make a transformation of this rectangle to a square centered at the origin having sides of length 2 (see figure 4).

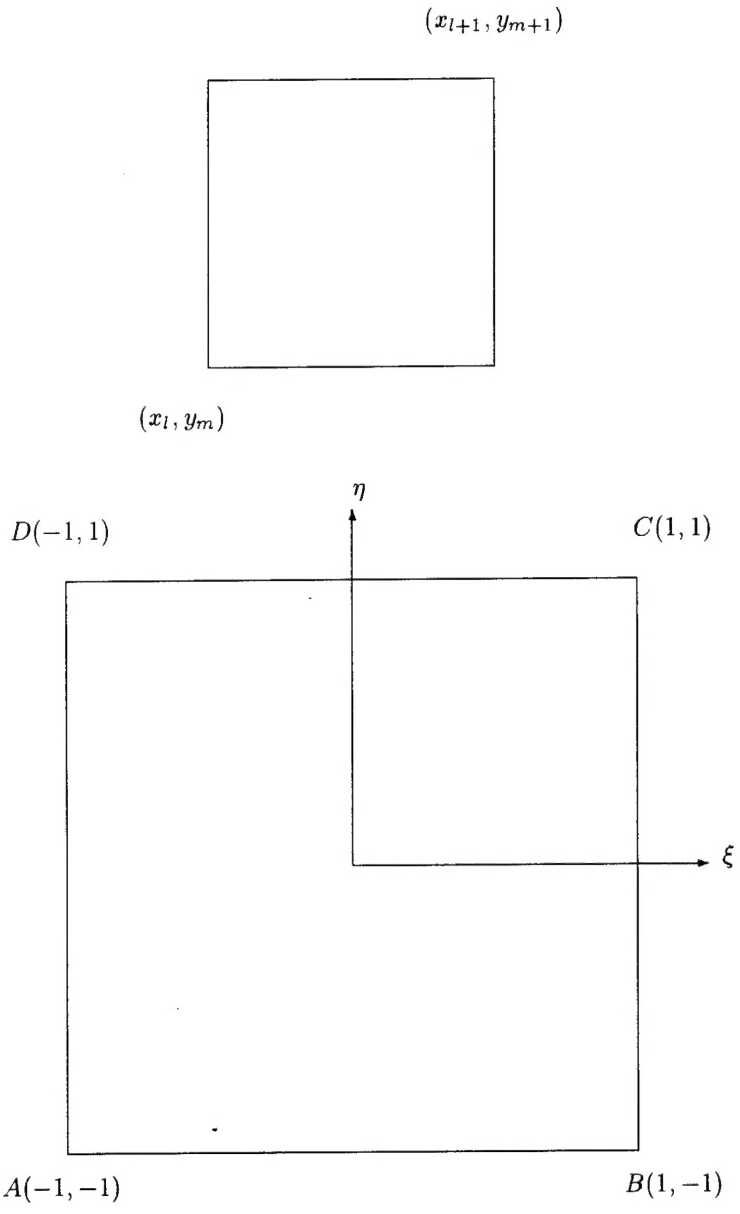

Figure 4: k^{th} element (top) and its transformed one

The transformation is given by

$$\begin{aligned} \xi &= \frac{2}{x_{l+1}-x_l}x - \frac{x_{l+1}+x_l}{x_{l+1}-x_l} \\ \eta &= \frac{2}{y_{m+1}-y_m}y - \frac{y_{m+1}+y_m}{y_{m+1}-y_m} \end{aligned} \tag{5}$$

and the basis functions in the $\xi - \eta$ domain are given by

$$\begin{aligned} \varphi_A &= \tfrac{1}{4}(\xi-1)(\eta-1) \\ \varphi_B &= -\tfrac{1}{4}(\xi+1)(\eta-1) \\ \varphi_C &= \tfrac{1}{4}(\xi+1)(\eta+1) \\ \varphi_D &= -\tfrac{1}{4}(\xi-1)(\eta+1) \end{aligned} \tag{6}$$

where the subscripts denote the vertex at which $\varphi = 1$. Note that the basis functions are product of the appropriate linear basis functions, i.e.

$$\varphi_A(x,y) = e_l(x)e_m(y)$$

where

$$e_i(\theta) = \frac{\theta_{i+1}-\theta}{\theta_{i+1}-\theta_i}.$$

This property is crucial to efficiently evaluating the integrals (Staniforth et al [7]).

The approximate problem becomes

$$M\dot{c} - Kc = b \tag{7}$$

where the entries of the matrices M, and K are given by

$$M_{ij} = \int\int_R \varphi_j\varphi_i dxdy \tag{8}$$

$$K_{ij} = \int\int_R \left(u\varphi_j\frac{\partial\varphi_i}{\partial x} + v\varphi_j\frac{\partial\varphi_i}{\partial y}\right)dxdy. \tag{9}$$

The vector c gives the concentrations at grid points at any time t, and b gives the boundary data

$$b_j = -\sum_{i=1}^{N_e} c_i(t)\left[\int_{y_L}^{y_U}(u\varphi_i\varphi_j)\,|_{x_L}^{x_R}dy + \int_{x_L}^{x_R}(v\varphi_i\varphi_j)\,|_{y_L}^{y_U}dx\right] \tag{10}$$

Since u, v are in general functions of x and y, we use numerical quadrature to evaluate K_{ij}. The quadrature we employed in our case is the two point open type, i.e.

$$\int_a^b f(x)dx = \frac{3h}{2}\left[f(a+h) + f(a+2h)\right], \tag{11}$$

where

$$h = \frac{b-a}{3}$$

and the error term is given by

$$\frac{3}{4}h^2 f''(\xi).$$

Thus for the first integral in K_{ij} we get

$$\int\int_R u\varphi_j \frac{\partial \varphi_i}{\partial x} dxdy = \sum_{k=1}^{N_e} \int_{x_l}^{x_{l+1}} \int_{y_m}^{y_{m+1}} u\varphi_j \frac{\partial \varphi_i}{\partial x} dxdy. \tag{12}$$

Now use the quadrature for each integral and centered differences for the partial derivatives to get

$$\begin{aligned} \textstyle\sum_{k=1}^{N_e} \frac{3}{2}h_x \frac{3}{2}h_y \Big\{ \quad & u(E)\varphi_j(E)\tfrac{\varphi_i(P)-\varphi_i(Q)}{2\delta} + \\ & u(H)\varphi_j(H)\tfrac{\varphi_i(Y)-\varphi_i(Z)}{2\delta} + \\ & u(F)\varphi_j(F)\tfrac{\varphi_i(V)-\varphi_i(S)}{2\delta} + \\ & u(G)\varphi_j(G)\tfrac{\varphi_i(W)-\varphi_i(X)}{2\delta} \Big\} \end{aligned} \tag{13}$$

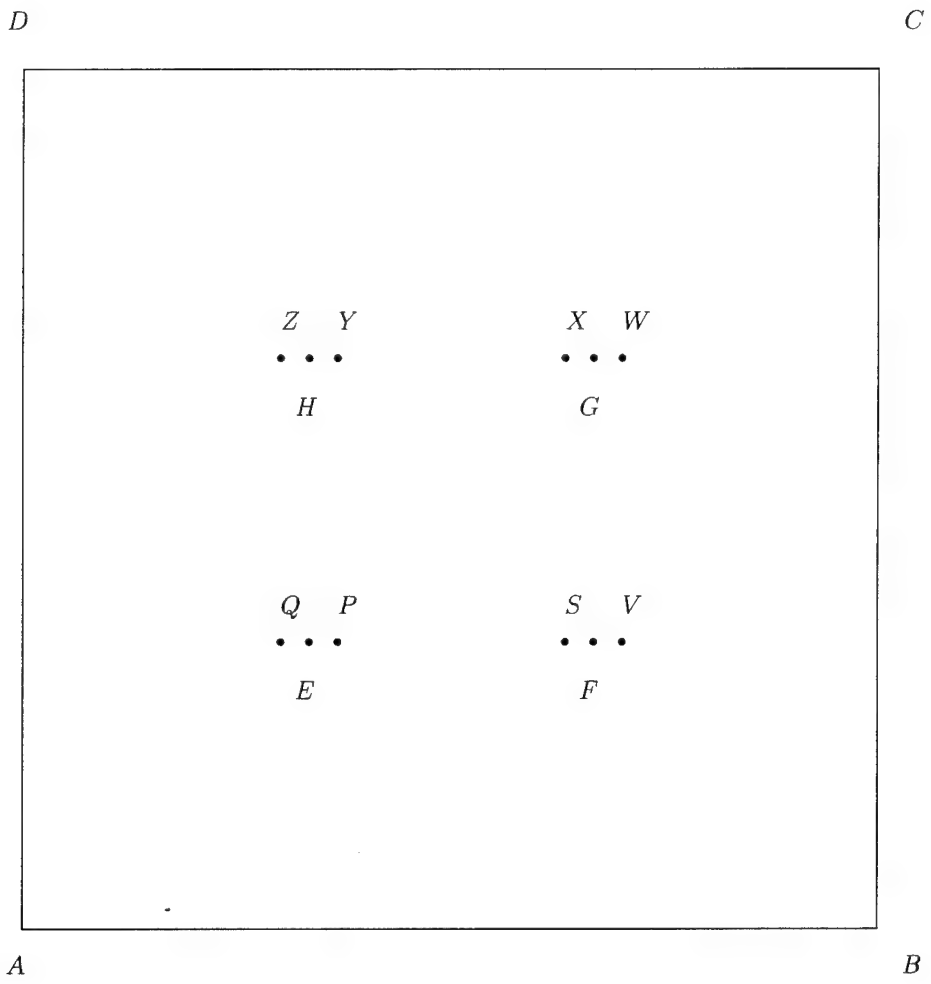

Figure 5: location of quadrature nodes

where

$$h_x = \frac{x_{l+1} - x_l}{3}, \qquad h_y = \frac{y_{m+1} - y_m}{3} \tag{14}$$

and δ, the spacing for the centered differences was arbitrarily chosen as

$$\delta = .05(x_{l+1} - x_l). \tag{15}$$

Similarly, we can approximate the second integral in K_{ij} except that now the points will be $\delta = .05(y_{m+1} - y_m)$ units above and below the four points E, F, G, H.

4. Semi-Lagrangian Finite Elements

Semi-Lagrangian schemes belong to the general class of upwinding methods. For hyperbolic equations, upwinding methods incorporate characteristic information into the numerical method. In Lagrangian schemes, the evolution of the system is monitored by following specific fluid particles through space. As a result, Lagrangian schemes allow larger time steps than Eulerian. The problem with fully Lagrangian schemes is that an initially regularly spaced set of particles will generally evolve into irregularly spaced particles. As a result, some important features in the flow may not be captured properly. Semi-Lagrangian schemes combine the best of both worlds: the regular resolution of an Eulerian scheme and the high stability of a Lagrangian method. The idea is to choose a different set of particles such that at the end of the time step, they arrive at points on a regular Cartesian grid. The departure points of the particles are determined by an iterative process using the interpolated velocity vector from the previous time.

A semi-Lagrangian formulation of (1)

$$\frac{c^+ - c^-}{2\Delta t} + \frac{1}{2}\left[(cu_x + cv_y)^+ (cu_x + cv_y)^-\right] = 0 \tag{16}$$

where c^+ is the solution at the grid points at time $t + \Delta t$, c^- is the solution at time $(t - \Delta t)$ at those points arriving at the grid points at time $t + \Delta t$. Since one requires two previous time levels, the program uses Matsuno's (see e.g. Haltiner and Williams [12]) method to get the first time step.

In the appendix, we bring plots of the solution for the cone test (see e.g. Zlatev [1]) using Eulerian finite elements with explicit, Crank-Nicholson and fully implicit time discretizations as well as the semi-Lagrangian method.

5. Stability Analysis

There are four rectangles having a vertex in common, as indicated in the next figure. The approximate solution at the vertices of the rectangles may be obtained by solving the following first order ordinary differential equation (see Neta and Williams [6] for the one dimensional advection case).

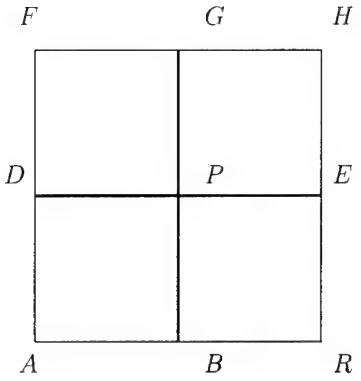

Figure 6: rectangular elements having a common vertex

$$\begin{aligned}
\dot{c}(P) \quad & +\tfrac{1}{4}\left[\dot{c}(G)+\dot{c}(B)+\dot{c}(D)+\dot{c}(E)\right] \\
& +\tfrac{1}{16}\left[\dot{c}(F)+\dot{c}(H)+\dot{c}(A)+\dot{c}(R)\right] \\
& +\tfrac{3}{16}u\tfrac{1}{\Delta x}\left\{c(H)-c(F)+c(R)-c(A)+4\left[c(E)-c(D)\right]\right\} \\
& +\tfrac{3}{16}v\tfrac{1}{\Delta y}\left\{c(H)-c(R)+c(F)-c(A)+4\left[c(G)-c(B)\right]\right\}=0
\end{aligned} \tag{17}$$

Substitute a Fourier mode

$$c(x,y,t)=A(t)e^{i(\mu x+\nu y)} \tag{18}$$

in (17) to get

$$\begin{aligned}
\dot{A}(t)\Big\{ \quad & 1+\tfrac{1}{4}(2\cos\mu\Delta x+2\cos\nu\Delta y)+\tfrac{1}{16}4\cos\mu\Delta x\cos\nu\Delta y\Big\} \\
& +\tfrac{3}{16}\tfrac{u}{\Delta x}A(t)\Big\{4+2\cos\nu\Delta y\Big\}2i\sin\mu\Delta x \\
& +\tfrac{3}{16}\tfrac{v}{\Delta v}A(t)\Big\{4+2\cos\mu\Delta x\Big\}2i\sin\nu\Delta y=0
\end{aligned} \tag{19}$$

or

$$\dot{A}(t) + 3i\left\{\frac{u}{\Delta x}\frac{\sin\mu\Delta x}{2+\cos\mu\Delta x} + \frac{v}{\Delta y}\frac{\sin\nu\Delta y}{2+\cos\nu\Delta y}\right\}A(t) = 0. \tag{20}$$

For the special case of flow along the x or y axis one of the terms in braces will drop. For flow along the diagonal

$$\frac{u}{v} = \frac{\Delta y}{\Delta x} \tag{21}$$

we have

$$\dot{A}(t) + 3i\frac{u}{\Delta x}\left\{\frac{\sin\mu\Delta x}{2+\cos\mu\Delta x} + \frac{\sin\nu\Delta y}{2+\cos\nu\Delta y}\right\}A(t) = 0. \tag{22}$$

In general, the ordinary differential equation becomes

$$\dot{A}(t) + i\sigma A(t) = 0, \tag{23}$$

where σ is 3 times the term in braces in (20). For the leap-frog time discretization

$$A_{n+1} - A_{n-1} + 2i\sigma\Delta t A_n = 0 \tag{24}$$

we have

$$\lambda_{1,2} = -i\sigma\Delta t \pm \sqrt{1-\sigma^2(\Delta t)^2}, \tag{25}$$

and thus for stability ($|\lambda| \leq 1$), we must have

$$|\sigma\Delta t| \leq 1. \tag{26}$$

If we let

$$\begin{aligned} U &= \max(|u|,|v|) \\ \delta &= \min(\Delta x, \Delta y) \end{aligned} \tag{27}$$

then the method is stable if

$$\frac{\Delta t}{\delta} \leq \frac{2}{3U}. \tag{28}$$

This is the CFL condition.

6. Fourier Transform

The Fourier transform of a function $c(x,y,t)$ is given by

$$\mathcal{F}\{c\}\,\hat{c}(k,l,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(x,y,t)e^{-i(kx+ly)}dxdy \tag{29}$$

Taking the Fourier transform of the linearization of (1), one gets the initial value problem

$$\frac{d\hat{c}}{dt} + i(ku+lv)\hat{c} = 0, \tag{30}$$

$$\hat{c}(k,l,0) = \hat{c}_0(k,l). \tag{31}$$

The solution of which is given by

$$\hat{c}(k,l,t) = \hat{c}_0(k,l)e^{i\nu t}, \tag{32}$$

where

$$\nu = -(ku + lv). \tag{33}$$

To get the solution $c(x,y,t)$ in the physical domain, we have to take the inverse Fourier transform and use the convolution theorem. This yields the well known solution

$$c(x,y,t) = c_0(x - ut, y - vt). \tag{34}$$

In order to obtain the Fourier transform of the approximate solution, recall that

$$\int_{-\infty}^{\infty} u(x + \Delta x, y, t)e^{-ikx}dx = e^{ik\Delta x}\hat{u}(k,y,t). \tag{35}$$

Applying Fourier transform to (17), one gets

$$\frac{\hat{c}}{dt} + \left[iu\frac{3}{\Delta x}\frac{\sin k\Delta x}{2 + \cos k\Delta x} + iv\frac{3}{\Delta y}\frac{\sin l\Delta y}{2 + \cos l\Delta y}\right]\hat{c} = 0. \tag{36}$$

Compare (36) and (30), to find that k, l are replaced by σ_x, σ_y respectively, where

$$\sigma_x = \frac{3}{\Delta x}\frac{\sin k\Delta x}{2 + \cos k\Delta x}, \qquad \sigma_y = \frac{3}{\Delta y}\frac{\sin l\Delta y}{2 + \cos l\Delta y}.$$

Note that as $\Delta x \to 0$, $\sigma_x \to k$ and as $\Delta y \to 0$, $\sigma_y \to l$, thus at the limit (36) becomes (30). In fact

$$\sigma_x \sim k - \frac{1}{180}k^5\Delta x^4 - \frac{1}{1512}k^7\Delta x^6 + \mathrm{O}\left(k^9\right)$$

$$\sigma_y \sim l - \frac{1}{180}l^5\Delta x^4 - \frac{1}{1512}l^7\Delta x^6 + \mathrm{O}\left(l^9\right)$$

The solution of (36) with the same initial value is given by

$$\hat{c}(k,l,t) = \hat{c}_0(k,l)e^{i\hat{\nu}t}, \tag{37}$$

where

$$\hat{\nu} = -(\sigma_x u + \sigma_y v). \tag{38}$$

The inverse transform is given by

$$c(x,y,t) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{c}_0(k,l)e^{-i(u\sigma_x + v\sigma_y)t}e^{i(kx+ly)}dkdl \tag{39}$$

or by using convolution

$$c(x,y,t) = c_0 * \mathcal{F}^{-1}\left\{e^{-i(u\sigma_x + v\sigma_y)t}\right\}. \tag{40}$$

7. Program Notes

The program can run semi-Lagrangian as well as Eulerian finite elements. In the Eulerian case the time differencing is one of the following:

- Explicit
- Crank Nicholson
- fully implicit

where the resulting linear system of equations is solved by the conjugate gradient method with symmetric SOR preconditioning (see e.g. Ortega [15]).

The four lines of input contain:

1. I, J
2. x_L,x_R, y_L, y_U
3. Δt, T (final time of integration), IPLOT (number of time steps between solution plots).
4. θ (a parameter dictating the time integrator). This is needed only for Eulerian finite elements.

Here we include the input file and the programs used to test the Eulerian and semi-Lagrangian finite element methods. First we give the input file. The first line contains the number of grid points in the x and y directions. The second line describe the rectangular domain on which the problem is solved. The interval for x is given followed by the interval for y. The third line gives the time step Δt, the final time of integration, and the number of time steps between plots. For Eulerian finite elements, we have a fourth line with the value of θ.

Here is the input file for the explicit time discretization.

```
21 21
0.0 2.0 0.0 2.0
0.0125 1.0 10
0.
```

Here is the program for the Eulerian finite elements.

```
*-----------------------------------------------------------------------*
*This program solves the 2D Advection Equation
*                  dc/dt + d/dx(cu) + d/dy(cv) = 0
*on a square domain using Periodic B.C's
*in both x and y using Bilinear Rectangular Finite Elements
*and THETA Time-Integration Algorithms.
* THETA=0   -> FORWARD EULER  (Explicit)
* THETA=1/2 -> CRANK-NICOLSON (Semi-Implicit)
* THETA=2/3 -> GALERKIN       (Semi-Implicit)
* THETA=1   -> BACKWARD EULER (Implicit)
*Written by F.X. Giraldo on 3/95
*           NRC Fellow
*           Department of Mathematics
*           Naval Postgraduate School
*           Monterey, CA 93940
*-----------------------------------------------------------------------*
      program fem_advect
      implicit real*8(a-h,o-z)
      parameter ( imax=21 )
c
c mxpoi  max number of points
c mxele  max number of elements
c mxbou  max number of boundary points
c nd     max number of vertices for each elements
c
      parameter ( mx=imax*imax, mxpoi=mx, mxele=mx, mxbou=mx/5, nd=4 )
c
c     global matrices
c
      dimension alhs(mxpoi,mxpoi), arhs(mxpoi,mxpoi), b(mxpoi)
      dimension coord(mxpoi,2)
      integer intma(mxele,nd), iboun(mxbou,4), node(imax,imax)
c
c     u velocity arrays
c
      dimension u(mxpoi)
c
c     v velocity arrays
c
      dimension v(mxpoi)
c
c     phi arrays
```

```
c
      dimension phip(mxpoi), phi0(mxpoi)
c
c     Read the Input Variables and create the Grid
c
c
c   input file contains 4 lines
c on first: number of grid points in x (nx) and y (ny) direction
c on second:range of x values (xmin, xmax),
c           range of y values (ymin,ymax)
c on third: delta t,
c            final time of integration (time_final),
c            number of times steps between plots (iplot)
c on fourth: theta (see above)
c
      call init(phi0,u,v,node,coord,intma,iboun,npoin,nelem,
     $          nboun,xmin,xmax,ymin,ymax,ym,nx,ny,nd,dx,dy,dt,
     $          ntime,theta,mxpoi,mxele,mxbou,imax,iplot)
      pi=4.0*atan(1.0)
      open(1,file='matlab.out')
c
c isets = number of time steps at which solution is plotted
c
      isets=ntime/iplot + 2
      write(1,*)isets
c
c  always plot initial condition
c
      call output(phi0,u,v,npoin,time,nx,ny,mxpoi)
c
c     begin the time marching
c
      time=0.0
c
c     CREATE THE STIFFNESS MATRIX ONCE
c           .
      call lhs(alhs,arhs,coord,intma,iboun,node,u,v,npoin,nelem,
     $         nboun,nx,ny,nd,dx,dy,dt,theta,mxpoi,mxele,mxbou,imax)
      do i=1,npoin
         do j=1,npoin
c            write(2,*)i,j,alhs(i,j)
         end do
      end do
c
```

```
c     TIME MARCH
c
      do itime=1,ntime

         time=time + dt
         write(*,'(" timestep time = ",i5,2x,e12.4)')itime,time/(2.0*pi)
c
c        Solve for the GeoPotential
c
         call rhs(b,arhs,phi0,coord,intma,iboun,node,
     $              npoin,nelem,nboun,nx,ny,nd,dx,dy,
     $              mxpoi,mxele,mxbou,imax)
         if (theta.eq.0.0) then
            call solve_explicit(alhs,phip,b,npoin,mxpoi)
         else if (theta.ne.0.0) then
            call pcgm(alhs,phip,b,npoin,mxpoi)
         endif
c
c        Enforce B.C.s Explicitly
c
         do i=1,nx
            j1=node(i,1)
            jny=node(i,ny)
            phip(j1)=phip(jny)
         end do

         do j=1,ny
            i1=node(1,j)
            inx=node(nx,j)
            phip(i1)=phip(inx)
         end do
c
c        Update
c
         do ip=1,npoin
            phi0(ip)=phip(ip)
         end do
c
c        check printing status
c
         if (mod(itime,iplot).eq.0)
     $      call output(phi0,u,v,npoin,time,nx,ny,mxpoi)

      end do
```

```
 1000 continue
      call output(phi0,u,v,npoin,time,nx,ny,mxpoi)
      close(1)
      stop
      end
*---------------------------------------------------------------------*
*This subroutine writes the output.  It is currently set only to
*print the concentration (or color) function at each node point.
*Written by F.X. Giraldo on 2/95
*---------------------------------------------------------------------*
      subroutine output(phi,u,v,npoin,time,nx,ny,mxpoi)
      implicit real*8(a-h,o-z)
      dimension phi(mxpoi), u(mxpoi), v(mxpoi)

      pi=4.0*atan(1.0)
      write(1,'(2(i6,1x),e16.8)'),nx,ny,time/(2.0*pi)
      write(1,'(e12.4)')(phi(ip), ip=1,npoin)
      return
      end
*---------------------------------------------------------------------*
*This subroutine reads in the input file.
*The info read is the number of grid points (in x and y), the domain,
*the time step, the final time, and the number of time steps for plotting.
*Written by F.X. Giraldo on 2/95
*---------------------------------------------------------------------*
      subroutine init(phi0,u0,v0,node,coord,intma,iboun,npoin,nelem,
     $                nboun,xmin,xmax,ymin,ymax,ym,nx,ny,nd,dx,dy,dt,
     $                ntime,theta,mxpoi,mxele,mxbou,imax,iplot)
      implicit real*8(a-h,o-z)
      dimension coord(mxpoi,2)
      dimension phi0(mxpoi), u0(mxpoi), v0(mxpoi)
      integer intma(mxele,nd), iboun(mxbou,4), node(imax,imax)
c
c     Read Input File
c
      read(*,*)nx,ny
      read(*,*)xmin,xmax,ymin,ymax
      read(*,*)dt,time_final,iplot
      read(*,*)theta
c
c     check bounds
c
      if (nx*ny.gt.mxpoi) then
```

```
         write(*,'(" Error! - Need to Enlarge MXPOI")')
         stop
      else if ((nx-1)*(ny-1).gt.mxele) then
         write(*,'(" Error! - Need to Enlarge MXELE")')
         stop
      else if (2*(nx-1)+2*(ny-1).gt.mxbou) then
         write(*,'(" Error! - Need to Enlarge MXBOU")')
         stop
      endif
c
c     set some constants
c
      pi=4.0*atan(1.0)
      ntime=nint(time_final/dt)
      dt=dt*2.0*pi
      time_final=time_final*2.0*pi
      xm=0.5*(xmax+xmin)
      ym=0.5*(ymax+ymin)
      dx=(xmax-xmin)/(nx-1)
      dy=(ymax-ymin)/(ny-1)
      xl=xmax-xmin
      yl=ymax-ymin
      w=1.0
      cx=0.25*xl
      cy=0.50*yl
      h=100.0
      rc=0.125*xl
      velmax=-1e5
c
c     set the Initial Conditions
c
      ip=0
      do j=1,ny
         y=ymin + real(j-1)*dy
         do i=1,nx
            x=xmin + real(i-1)*dx
            ip=ip+1
            r=sqrt( (x-cx)**2 + (y-cy)**2 )
            phi0(ip)=0.0
            if (r.lt.rc) then
               phi0(ip)=h*(1.0-r/rc)
            endif
            u0(ip)=+(y-ym)
            v0(ip)=-(x-xm)
```

```
            vel1=u0(ip)**2 + v0(ip)**2
            velmax=max(velmax,vel1)
         end do
      end do
      cfl=dt*sqrt(velmax/(dx**2 + dy**2))
      print*,' ** CFL = ',cfl
c
c     GENERATE COORD
c
      npoin=nx*ny
      ip=0
      do j=1,ny
         do i=1,nx
            ip=ip+1
            node(i,j)=ip
            coord(ip,1)=xi + dx*real(i-1)
            coord(ip,2)=yi + dy*real(j-1)
         end do
      end do
c
c     GENERATE INTMA
c
      nelem=(nx-1)*(ny-1)
      ie=0
      do j=1,ny-1
         do i=1,nx-1
            ie=ie+1
            intma(ie,1)=node(i,j)
            intma(ie,2)=node(i+1,j)
            intma(ie,3)=node(i+1,j+1)
            intma(ie,4)=node(i,j+1)
         end do
      end do
c
c     GENERATE IBOUN
c
      nboun=2*(nx-1) + 2*(ny-1)
      ib=0

c     bottom (y=ymin)
      ie=1
      do i=1,nx-1
         ib=ib+1
         iboun(ib,1)=node(i,1)
```

```
         iboun(ib,2)=node(i+1,1)
         iboun(ib,3)=ie
         iboun(ib,4)=1
         ie=ie+1
      end do

c     top (y=ymax)
      ie=nelem
      do i=nx,2,-1
         ib=ib+1
         iboun(ib,1)=node(i,ny)
         iboun(ib,2)=node(i-1,ny)
         iboun(ib,3)=ie
         iboun(ib,4)=1
         ie=ie-1
      end do

c     right (x=xmax)
      ie=(nx-1)
      do j=1,ny-1
         ib=ib+1
         iboun(ib,1)=node(nx,j)
         iboun(ib,2)=node(nx,j+1)
         iboun(ib,3)=ie
         iboun(ib,4)=2
         ie=ie + (nx-1)
      end do

c     left (x=xmin)
      ie=nelem - (nx-1) + 1
      do j=ny,2,-1
         ib=ib+1
         iboun(ib,1)=node(1,j)
         iboun(ib,2)=node(1,j-1)
         iboun(ib,3)=ie
         iboun(ib,4)=2
         ie=ie - (nx-1)
      end do

      return
      end
*-----------------------------------------------------------------------*
*This subroutine constructs the LHS matrix for Bilinear Rectangular
*Elements for the Advection Equation with Periodic
```

```
*East-West and North-South Boundary Conditions.
*Written by F.X. Giraldo on 2/95
*-----------------------------------------------------------------*
      subroutine lhs(alhs,arhs,coord,intma,iboun,node,u,v,npoin,nelem,
     $           nboun,nx,ny,nd,dx,dy,dt,theta,mxpoi,mxele,mxbou,imax)
      implicit real*8(a-h,o-z)
      parameter (mx=3000)
c
c     global arrays
c
      dimension alhs(mxpoi,mxpoi), arhs(mxpoi,mxpoi)
      dimension coord(mxpoi,2), u(mxpoi), v(mxpoi)
      integer intma(mxele,nd), node(imax,imax), iboun(mxbou,4)
c
c     local coordinate system for a CCW ordered rectangle
c
      dimension xi(4), eta(4), a_temp(mx)
      data xi  /-1, 1, 1,-1 /
      data eta /-1,-1, 1, 1 /
c
c     initialize the global matrix
c
      do j=1,npoin
do i=1,npoin
   alhs(i,j)=0.0
   arhs(i,j)=0.0
        end do
      end do
c
c     loop thru the elements
c
      do ie=1,nelem
c
c     assemble element matrix  cm  == consistent mass matrix
c                              ckc == conduction-like matrix
         do i=1,nd
            ii=intma(ie,i)
            cm_lump=0.0
            do j=1,nd
               jj=intma(ie,j)
               cm=(2.0+2.0/3.0*xi(i)*xi(j))*(2.0+2.0/3.0*eta(i)*eta(j))
               ckx=0.0
               cky=0.0
               do k=1,nd
```

```
                uk=u(intma(ie,k))
                vk=v(intma(ie,k))
                ckx=ckx + uk*
     $  (2.0*xi(i) + 2.0/3.0*xi(i)*xi(j)*xi(k))*
     $  (2.0+2.0/3.0*(eta(i)*eta(j) + eta(i)*eta(k) + eta(j)*eta(k)))
                cky=cky + vk*
     $  (2.0*eta(i) + 2.0/3.0*eta(i)*eta(j)*eta(k))*
     $  (2.0+2.0/3.0*(xi(i)*xi(j) + xi(i)*xi(k) + xi(j)*xi(k)))
              end do
              cmm=dx*dy/64.0*cm
              ck=dy/128.0*ckx + dx/128.0*cky
              alhs(ii,jj)=alhs(ii,jj) + 1.0/dt*cmm - theta*ck
              arhs(ii,jj)=arhs(ii,jj) + 1.0/dt*cmm + (1.0-theta)*ck
           end do
        end do
     end do
c
c    Account for Periodicity
c
     do i=1,nx
        j1=node(i,1)
        jny=node(i,ny)
        do jj=1,npoin
           a_temp(jj)=alhs(j1,jj) + alhs(jny,jj)
        end do
        do jj=1,npoin
           alhs(j1,jj)=a_temp(jj)
           alhs(jny,jj)=a_temp(jj)
        end do
        alhs(j1,j1)=a_temp(j1) + a_temp(jny)
        alhs(jny,jny)=a_temp(j1) + a_temp(jny)
     end do

     do j=1,ny
        i1=node(1,j)
        inx=node(nx,j)
        do jj=1,npoin
           a_temp(jj)=alhs(i1,jj) + alhs(inx,jj)
        end do
        do jj=1,npoin
           alhs(i1,jj)=a_temp(jj)
           alhs(inx,jj)=a_temp(jj)
        end do
        alhs(i1,i1)=a_temp(i1) + a_temp(inx)
```

```
         alhs(inx,inx)=a_temp(i1) + a_temp(inx)
      end do
c
c     If theta=0, then Lump.
c
      if (theta.eq.0.0) then
         do i=1,npoin
            asum=0.0
            do j=1,npoin
               asum=asum + alhs(i,j)
               alhs(i,j)=0.0
            end do
            alhs(i,i)=asum
         end do
      endif

      return
      end
*-------------------------------------------------------------------*
*The next 4 Subroutines solve a linear system [A]{x}={b}, for n unknowns using
*the CONJUGATE GRADIENT METHOD with an SSOR preconditioner.
*The variable w determines the preconditioner. for w=1 it is
*Symmetric Jacobi and for w>1 it is Symmetric SOR (SSOR).
*Written by F.X. Giraldo on 4/14/90
*Modified for any type of spd Matrix A on 2/95
*-------------------------------------------------------------------*
      subroutine pcgm(a,x,b,n,mx)
      implicit real*8(a-h,o-z)
      parameter (nmax=3000, kmax=15, w=1.0)
      dimension a(mx,mx), b(mx), x(mx)
      dimension r(nmax), rw(nmax), p(nmax), ap(nmax)

      do i=1,n
         sum=0.0
         do j=1,n
            sum=sum + a(i,j)*x(j)
         end do
         r(i)=b(i) - sum
      end do

      call ssor(r,rw,a,n,w,nmax,mx)

      do i=1,n
         p(i)=rw(i)
```

```
      end do

      call ip(rop,rw,r,n,nmax)

      do k=1,kmax

         do i=1,n
            sum=0.0
            do j=1,n
               sum=sum + a(i,j)*p(j)
            end do
            ap(i)=sum
         end do

         alfden=0.0
         do i=1,n
            alfden=alfden + p(i)*ap(i)
         end do
         alf=-rop/alfden

         do i=1,n
            x(i)=x(i)-alf*p(i)
         end do

         do i=1,n
            r(i)=r(i) + alf*ap(i)
         end do

         call convtest(rop,r,n,flag,nmax)
         if (flag .eq. 1.0) goto 200

         call ssor(r,rw,a,n,w,nmax,mx)
         call ip(rnp,rw,r,n,nmax)

         beta=rnp/rop
         do i=1,n
            p(i)=rw(i) + beta*p(i)
         end do
         rop=rnp
      end do
      k=k-1
      print*,'      No Convergence'
200   continue
```

```
      print*,'      SSOR PCG Loops = ',k
      return
      end
*-----------------------------------------------------------------------*
      subroutine ssor(r,rw,a,n,w,nmax,mx)
      implicit real*8(a-h,o-z)
      dimension r(nmax), rw(nmax)
      dimension a(mx,mx)

* symmetric sor on the residual
* zeroing the wiggle residual

      do i=1,n
         rw(i)=0.0
      end do

* up sweep

      do i=1,n
         sum=0.0
         do j=1,n
            sum=sum + a(i,j)*rw(j)
         end do
         rw(i)=rw(i) + w/a(i,i)*( r(i) - sum )
      end do

* down sweep

      do i=n,1,-1
         sum=0.0
         do j=1,n
            sum=sum + a(i,j)*rw(j)
         end do
         rw(i)=rw(i) + w/a(i,i)*( r(i) - sum )
      end do
      return
      end
*-----------------------------------------------------------------------*
      subroutine ip(f,fw,fr,n,nmax)
      implicit real*8(a-h,o-z)
      dimension fw(nmax), fr(nmax)
      f=0.0
      do i=1,n
         f=f + fw(i)*fr(i)
```

```
      end do
      return
      end
*-----------------------------------------------------------------------*
      subroutine convtest(rop,r,n,flag,nmax)
      implicit real*8(a-h,o-z)
      dimension r(nmax)

      emin=1.0e-6
      flag=0.0
      rtest=0.0
      if (rop .lt. emin) then
         do i=1,n
            rtest=rtest + r(i)**2.0
         end do
         if (rtest .lt. emin) flag=1.0
      endif
      return
      end
*-----------------------------------------------------------------------*
*This subroutine Builds the RHS vector for Bilinear Rectangular Finite
*Elements for the 2D SLSI Shallow Water Equations with Periodic
*West-East and North-South Boundaries.
*Written by F.X. Giraldo on 2/95
*-----------------------------------------------------------------------*
      subroutine rhs(b,arhs,phi0,coord,intma,iboun,node,
     $               npoin,nelem,nboun,nx,ny,nd,dx,dy,
     $               mxpoi,mxele,mxbou,imax)
      implicit real*8(a-h,o-z)
      parameter (mx=3000)
c
c     global arrays
c
      dimension b(mxpoi), arhs(mxpoi,mxpoi), phi0(mxpoi)
      dimension coord(mxpoi,2)
      integer intma(mxele,nd), iboun(mxbou,4), node(imax,imax)
c
c     local coordinates for a CCW ordered Rectangle
c
      dimension xi(4), eta(4)
      data xi  /-1, 1, 1,-1/
      data eta /-1,-1, 1, 1/
c
c     initialize the right hand side
```

```
c
      do ip=1,npoin
b(ip)=0.0
      end do
c
c     Construct the RHS Vector
c
      do i=1,npoin
         do j=1,npoin
            b(i)=b(i) + arhs(i,j)*phi0(j)
         end do
      end do
c
c     Account for Periodicity
c
      do i=1,nx
         j1=node(i,1)
         jny=node(i,ny)
         b_temp=b(j1) + b(jny)
         b(j1)=b_temp
         b(jny)=b_temp
      end do
      do j=1,ny
         i1=node(1,j)
         inx=node(nx,j)
         b_temp=b(i1) + b(inx)
         b(i1)=b_temp
         b(inx)=b_temp
      end do
      return
      end
*-----------------------------------------------------------------------*
*This subroutine solves a Linear NxN system where the Coefficient
*Matrix is diagonal.
*Written by F.X. Giraldo on 4/14/90
*-----------------------------------------------------------------------*
      subroutine solve_explicit(a,x,b,n,mx)
      implicit real*8(a-h,o-z)
      dimension a(mx,mx), b(mx), x(mx)
      do i=1,n
         x(i)=b(i)/a(i,i)
      end do
      return
      end
```

Here is the program for the semi-Lagrangian finite elements.

```
*-----------------------------------------------------------------------*
*This program solves the 2D Advection Equation
*                  dc/dt + d/dx(cu) + d/dy(cv) = 0
*on a square domain using Periodic B.C.'s
*in both x and y and using Semi-Implicit Semi-Lagrangian
*Bilinear Rectangular Finite Elements.
*Written by F.X. Giraldo on 3/95
*           NRC Fellow
*           Department of Mathematics
*           Naval Postgraduate School
*           Monterey, CA 93940
*-----------------------------------------------------------------------*
      program slt_advect
      implicit real*8(a-h,o-z)
      parameter ( imax=21 )
c
c mxpoi  max number of points
c mxele  max number of elements
c mxbou  max number of boundary points
c nd     max number of vertices for each elements
c
      parameter ( mx=imax*imax, mxpoi=mx, mxele=mx, mxbou=mx/5, nd=4 )
c
c     global matrices
c
      dimension a(mxpoi,mxpoi), b(mxpoi)
      dimension f(mxpoi)
      dimension coord(mxpoi,2)
      dimension cmat(mxpoi)
      integer intma(mxele,nd), iboun(mxbou,4)
c
c     u velocity arrays
c
      dimension um(mxpoi),      u0(mxpoi),     up(mxpoi)
      dimension                 u0_x(mxpoi)
c
c     v velocity arrays
c
      dimension vm(mxpoi),      v0(mxpoi),     vp(mxpoi)
      dimension                 v0_y(mxpoi)
c
c     phi arrays
```

```
c
      dimension phim(mxpoi),    phi0(mxpoi),   phip(mxpoi)
c
c     departure point arrays
c
      dimension alfm(mxpoi,2),  alf0(mxpoi,2)
c
c     spline derivative arrays
c
      dimension dphim(mxpoi)
      dimension du0(mxpoi), du0_x(mxpoi)
      dimension dv0(mxpoi), dv0_y(mxpoi)
c
c     Auxiliary Matrices
c
      integer node(imax,imax)
c
c     Read the Input Variables and create the Grid
c
c   input file contains 4 lines
c on first: number of grid points in x (nx) and y (ny) direction
c on second:range of x values (xmin, xmax),
c           range of y values (ymin,ymax)
c on third: delta t,
c            final time of integration (time_final),
c            number of times steps between plots (iplot)
c
      call init(phi0,u0,v0,node,coord,alf0,intma,iboun,npoin,
     $          nelem,nboun,xmin,xmax,ymin,ymax,nx,ny,dx,dy,dt,
     $          ntime,ym,nd,mxpoi,mxele,mxbou,imax,iplot)
c
c     Construct the Lumped Mass Matrix used for the derivative computation
c
      call get_geom(nelem,npoin,intma,coord,cmat,node,
     $              nx,ny,nd,dx,dy,mxpoi,mxele,imax)
      time=0.0
      pi=4.0*atan(1.0)
      open(1,file='matlab.out')
c
c isets = number of time steps at which solution is plotted
c
      isets=ntime/iplot + 2
      write(1,*)isets
c
```

```
c  always plot initial condition
c
      call output(phi0,u0,v0,npoin,time,nx,ny,mxpoi)
c
c     begin the time marching
c
c     Do the 1st Time-step Integration to get 2-time levels
c
      do itime=1,1
         time=time + dt
         dtime=time/(2.0*pi)
         write(*,'(" timestep time = ",i5,2x,e12.4)')itime,dtime
         call matsuno(phim,phi0,phip,um,u0,up,
     $                vm,v0,vp,coord,intma,npoin,nelem,dt,dx,dy,
     $                node,nx,ny,ym,nd,mxpoi,mxele,imax)
         call update(phim,phi0,phip,um,u0,up,vm,v0,vp,alfm,alf0,
     $               npoin,mxpoi)
      end do
c
c     TIME MARCH
c
      do itime=2,ntime

         time=time + dt
         dtime=time/(2.0*pi)
         write(*,'(" timestep time = ",i5,2x,e12.4)')itime,dtime
*
*        1st, DETERMINE DEPARTURE POINT
*
         call splie2(du0,u0,coord,nx,ny,node,mxpoi,imax)
         call splie2(dv0,v0,coord,nx,ny,node,mxpoi,imax)
         call depart(alf0,alfm,coord,u0,du0,v0,dv0,
     $               npoin,dt,xmin,xmax,ymin,ymax,
     $               nd,mxpoi,node,nx,ny,imax)
*
*        2nd, COMPUTE DERIVATIVES
*
         call deriv_x(u0_x,u0,intma,cmat,node,npoin,nelem,
     $                nx,ny,nd,dy,mxpoi,mxele,imax)
         call deriv_y(v0_y,v0,intma,cmat,node,npoin,nelem,
     $                nx,ny,nd,dx,mxpoi,mxele,imax)
*
*        3rd, INTERPOLATE PHIM, UM_X=U0_X, VM_Y=V0_Y
*        at the departure point = X - 2*ALPHA
```

```
*
         call splie2(dphim,phim,coord,nx,ny,node,mxpoi,imax)
         call splie2(du0_x,u0_x,coord,nx,ny,node,mxpoi,imax)
         call splie2(dv0_y,v0_y,coord,nx,ny,node,mxpoi,imax)
c
c        INTERPOLATE THE RIGHT HAND SIDE FUNCTION
c
         call interp(f,phim,u0_x,v0_y,dphim,du0_x,dv0_y,coord,
     $               alf0,node,npoin,dt,xmin,xmax,ymin,ymax,ym,
     $               nx,ny,nd,mxpoi,imax)
         do ip=1,npoin
            phip(ip)=f(ip)
            up(ip)=u0(ip)
            vp(ip)=v0(ip)
         end do
c
c        Update the values
c
         call update(phim,phi0,phip,um,u0,up,vm,v0,vp,alfm,alf0,
     $               npoin,mxpoi)
c
c     check time for printing output
c
         if (mod(itime,iplot).eq.0)
     $      call output(phi0,u0,v0,npoin,time,nx,ny,mxpoi)

      end do

 1000 continue

      call output(phi0,u0,v0,npoin,time,nx,ny,mxpoi)
      close(1)
      stop
      end
*-----------------------------------------------------------------------*
*This subroutine finds the departure point
* ALPHA1=DT*U(X-ALPHA1,Y-ALPHA2,T) ALPHA2=DT*V(X-ALPHA1,Y-ALPHA2,T)
*Written by F.X. Giraldo on 2/95
*-----------------------------------------------------------------------*
      subroutine depart(alf0,alfm,coord,u0,du0,v0,dv0,
     $                  npoin,dt,xmin,xmax,ymin,ymax,
     $                  nd,mxpoi,node,nx,ny,imax)
      implicit real*8(a-h,o-z)
      dimension alf0(mxpoi,2), alfm(mxpoi,2)
```

```
      dimension u0(mxpoi), du0(mxpoi)
      dimension v0(mxpoi), dv0(mxpoi)
      dimension coord(mxpoi,2)
      integer node(imax,imax)

      do ip=1,npoin
         alpha1=alfm(ip,1)
         alpha2=alfm(ip,2)
         do k=1,3
            xd=coord(ip,1) - alpha1
            yd=coord(ip,2) - alpha2
            if (xd.lt.xmin) xd=xmax - (xmin - xd)
            if (xd.gt.xmax) xd=xmin + (xd - xmax)
            if (yd.lt.ymin) yd=ymax - (ymin - yd)
            if (yd.gt.ymax) yd=ymin + (yd - ymax)

            if ( (xd.lt.xmin.or.xd.gt.xmax) .or.
     $           (yd.lt.ymin.or.yd.gt.ymax) ) then
               print*,' Error in DEPART'
               print*,'XD out of Range = ',xd,yd
               print*,'ip alpha = ',ip,alpha1,alpha2
               stop
            endif

            call splin2(u,coord,u0,du0,xd,yd,nx,ny,node,mxpoi,imax)
            call splin2(v,coord,v0,dv0,xd,yd,nx,ny,node,mxpoi,imax)

            alpha1=dt*u
            alpha2=dt*v
         end do
         alf0(ip,1)=alpha1
         alf0(ip,2)=alpha2
      end do

      return
      end
*-----------------------------------------------------------------------*
*This subroutine computes the 4th Order Accurate derivative WRT X of the
*variable UNKNO and stores it in DERIP using Bilinear Rectangular
*Finite Elements
*Written by F.X. Giraldo on 2/95
*-----------------------------------------------------------------------*
      subroutine deriv_x(derip,unkno,intma,cmat,node,npoin,nelem,
     $                   nx,ny,nd,dy,mxpoi,mxele,imax)
```

```
      implicit real*8(a-h,o-z)
c
c     global arrays
c
      dimension derip(mxpoi), unkno(mxpoi)
      dimension cmat(mxpoi)
      integer intma(mxele,nd), node(imax,imax)
c
c     local coordinate system for a CCW ordered rectangle
c
      dimension xi(4), eta(4)
      data xi  / -1, 1, 1,-1 /
      data eta / -1,-1, 1, 1 /
c
c     initialize arrays
c
      do ip=1,npoin
         derip(ip)=0.0
      end do
c
c     Loop thru the Elements and find the 1St DERIVATIVES
c
      do ie=1,nelem
         do i=1,nd
            phi_x=0.0
            ip=intma(ie,i)
            do k=1,nd
               phik=unkno(intma(ie,k))
               phi_x=phi_x + xi(k)*phik*( 2.0 + 2.0/3.0*eta(k)*eta(i) )
            end do
            derip(ip)=derip(ip) + dy/16.0*phi_x
         end do
      end do
c
c     Account for Periodicity
c
      do j=1,ny
         i1=node(1,j)
         inx=node(nx,j)
         derip_temp=derip(i1) + derip(inx)
         derip(i1)=derip_temp
         derip(inx)=derip_temp
      end do
c
```

```
c     Now multiply by the Inverse of the Lumped mass matrix
c
      do ip=1,npoin
         derip(ip)=derip(ip)*cmat(ip)
      end do

      return
      end
*---------------------------------------------------------------------*
*This subroutine computes the 4th Order Accurate derivative WRT Y of the
*variable UNKNO and stores it in DERIP using Bilinear Rectangular
*Finite Elements
*Written by F.X. Giraldo on 2/95
*---------------------------------------------------------------------*
      subroutine deriv_y(derip,unkno,intma,cmat,node,npoin,nelem,
     $                   nx,ny,nd,dx,mxpoi,mxele,imax)
      implicit real*8(a-h,o-z)
c
c     global arrays
c
      dimension derip(mxpoi), unkno(mxpoi)
      dimension cmat(mxpoi)
      integer intma(mxele,nd), node(imax,imax)
c
c     local coordinate system for a CCW ordered rectangle
c
      dimension xi(4), eta(4)
      data xi  / -1, 1, 1,-1 /
      data eta / -1,-1, 1, 1 /
c
c     initialize arrays
c
      do ip=1,npoin
         derip(ip)=0.0
      end do
c
c     Loop thru the Elements and find the 1St DERIVATIVES
c
      do ie=1,nelem
         do i=1,nd
            phi_y=0.0
            ip=intma(ie,i)
            do k=1,nd
               phik=unkno(intma(ie,k))
```

```
            phi_y=phi_y + eta(k)*phik*( 2.0 + 2.0/3.0*xi(k)*xi(i) )
          end do
          derip(ip)=derip(ip) + dx/16.0*phi_y
        end do
      end do
c
c     Account for Periodicity
c
      do j=1,ny
        i1=node(1,j)
        inx=node(nx,j)
        derip_temp=derip(i1) + derip(inx)
        derip(i1)=derip_temp
        derip(inx)=derip_temp
      end do
c
c     Now multiply by the Inverse of the Lumped mass matrix
c
      do ip=1,npoin
        derip(ip)=derip(ip)*cmat(ip)
      end do

      return
      end
*-----------------------------------------------------------------------*
*This subroutine computes the Inverse Lumped Mass Matrix
*for Bilinear Rectangular Finite Elements used for obtaining the
*4th Order Accurate 1st and 2nd derivatives in both X and Y.
*Written by F.X. Giraldo on 2/95
*-----------------------------------------------------------------------*
      subroutine get_geom(nelem,npoin,intma,coord,cmat,node,
     $                    nx,ny,nd,dx,dy,mxpoi,mxele,imax)
      implicit real*8(a-h,o-z)
c
c     global arrays
c
      dimension coord(mxpoi,2), cmat(mxpoi)
      integer intma(mxele,nd), node(imax,imax)
c
c     Compute the inverse lumped mass matrix
c
      do ip=1,npoin
        cmat(ip)=0.0
      end do
```

```
      do ie=1,nelem
         do in=1,nd
            ip=intma(ie,in)
            cmat(ip)=cmat(ip) + dx*dy/4.0
         end do
      end do
c
c     Account for Periodicity in X
c
      do j=1,ny
         i1=node(1,j)
         inx=node(nx,j)
         cmat_temp=cmat(i1) + cmat(inx)
         cmat(i1)=cmat_temp
         cmat(inx)=cmat_temp
      end do
c
c     Invert the lumped mass matrix
c
      do ip=1,npoin
         cmat(ip)=1.0/cmat(ip)
      end do

      return
      end
*---------------------------------------------------------------------*
*This subroutine reads in the input file.
*The info read is the number of grid points (in x and y), the domain,
*the time step, the final time, and the number of time steps for plotting.
*Written by F.X. Giraldo on 2/95
*---------------------------------------------------------------------*
      subroutine init(phi0,u0,v0,node,coord,alf0,intma,iboun,npoin,
     $                nelem,nboun,xmin,xmax,ymin,ymax,nx,ny,dx,dy,dt,
     $                ntime,ym,nd,mxpoi,mxele,mxbou,imax,iplot)
      implicit real*8(a-h,o-z)
      dimension coord(mxpoi,2), alf0(mxpoi,2)
      dimension phi0(mxpoi), u0(mxpoi), v0(mxpoi)
      integer intma(mxele,nd), iboun(mxbou,4), node(imax,imax)
c
c     Read Input File
c
      read(*,*)nx,ny
      read(*,*)xmin,xmax,ymin,ymax
```

```
      read(*,*)dt,time_final,iplot
c
c     check bounds
c
      if (nx*ny.gt.mxpoi) then
         write(*,'(" Error! - Need to Enlarge MXPOI")')
         stop
      else if ((nx-1)*(ny-1).gt.mxele) then
         write(*,'(" Error! - Need to Enlarge MXELE")')
         stop
      else if (2*(nx-1)+2*(ny-1).gt.mxbou) then
         write(*,'(" Error! - Need to Enlarge MXBOU")')
         stop
      endif
c
c     set some constants
c
      pi=4.0*atan(1.0)
      ntime=nint(time_final/dt)
      dt=dt*2.0*pi
      time_final=time_final*2.0*pi
      xm=0.5*(xmax+xmin)
      ym=0.5*(ymax+ymin)
      dx=(xmax-xmin)/(nx-1)
      dy=(ymax-ymin)/(ny-1)
      xl=xmax-xmin
      yl=ymax-ymin
      w=1.0
      cx=0.25*xl
      cy=0.50*yl
      h=100.0
      rc=0.125*xl
      velmax=-1e5
c
c     set the Initial Conditions
c
      ip=0
      do j=1,ny
         y=ymin + real(j-1)*dy
         do i=1,nx
            x=xmin + real(i-1)*dx
            ip=ip+1
            r=sqrt( (x-cx)**2 + (y-cy)**2 )
            phi0(ip)=0.0
```

```
            if (r.lt.rc) then
               phi0(ip)=h*(1.0-r/rc)
            endif
            u0(ip)=+(y-ym)
            v0(ip)=-(x-xm)
            vel1=u0(ip)**2 + v0(ip)**2
            velmax=max(velmax,vel1)
         end do
      end do
      cfl=dt*sqrt(velmax/(dx**2 + dy**2))
      print*,' ** CFL = ',cfl
c
c     GENERATE COORD
c
      npoin=nx*ny
      ip=0
      do j=1,ny
         do i=1,nx
            ip=ip+1
            node(i,j)=ip
            coord(ip,1)=xi + dx*real(i-1)
            coord(ip,2)=yi + dy*real(j-1)
            alf0(ip,1)=0.5*dx
            alf0(ip,2)=0.5*dy
         end do
      end do
c
c     GENERATE INTMA
c
      nelem=(nx-1)*(ny-1)
      ie=0
      do j=1,ny-1
         do i=1,nx-1
            ie=ie+1
            intma(ie,1)=node(i,j)
            intma(ie,2)=node(i+1,j)
            intma(ie,3)=node(i+1,j+1)
            intma(ie,4)=node(i,j+1)
         end do
      end do
c
c     GENERATE IBOUN
c
      nboun=2*(nx-1)
```

```
      ib=0

c     bottom (y=ymin)
      ie=1
      do i=1,nx-1
         ib=ib+1
         iboun(ib,1)=node(i,1)
         iboun(ib,2)=node(i+1,1)
         iboun(ib,3)=ie
         iboun(ib,4)=3
         ie=ie+1
      end do

c     top (y=ymax)
      ielem=nelem
      do i=nx,2,-1
         ib=ib+1
         iboun(ib,1)=node(i,ny)
         iboun(ib,2)=node(i-1,ny)
         iboun(ib,3)=ielem
         iboun(ib,4)=3
         ielem=ielem-1
      end do

      return
      end
*-------------------------------------------------------------------*
*This subroutine interpolates the right hand side function for the
*2D Advection Equation using a 3-time level Semi-Lagrangian
*Semi-Implicit Method
*Written by F.X. Giraldo on 2/95
*-------------------------------------------------------------------*
      subroutine interp(f,phim,u0_x,v0_y,dphim,du0_x,dv0_y,coord,
     $               alf0,node,npoin,dt,xmin,xmax,ymin,ymax,ym,
     $               nx,ny,nd,mxpoi,imax)
      implicit real*8(a-h,o-z)
      dimension f(mxpoi)
      dimension phim(mxpoi), u0_x(mxpoi), v0_y(mxpoi)
      dimension dphim(mxpoi),du0_x(mxpoi),dv0_y(mxpoi)
      dimension coord(mxpoi,2), alf0(mxpoi,2)
      integer node(imax,imax)

      do ip=1,npoin
c
```

```
c     1st, Interpolate "-" values = F( x - 2*alpha, t - dt )
c
         xd=coord(ip,1) - 2.0*alf0(ip,1)
         yd=coord(ip,2) - 2.0*alf0(ip,2)
         if (xd.lt.xmin) xd=xmax - (xmin - xd)
         if (xd.gt.xmax) xd=xmin + (xd - xmax)
         if (yd.lt.ymin) yd=ymax - (ymin - yd)
         if (yd.gt.ymax) yd=ymin + (yd - ymax)

         if ( (xd.lt.xmin.or.xd.gt.xmax) .or.
     $        (yd.lt.ymin.or.yd.gt.ymax) ) then
            print*,' Error in INTERP'
            print*,'XD out of Range = ',xd,yd
            stop
         endif

c        Do Interpolation
         call splin2(phi,coord,phim,dphim,xd,yd,nx,ny,node,mxpoi,imax)
         call splin2(u_x,coord,u0_x,du0_x,xd,yd,nx,ny,node,mxpoi,imax)
         call splin2(v_y,coord,v0_y,dv0_y,xd,yd,nx,ny,node,mxpoi,imax)

         f(ip)=(1.0-dt*(u_x + v_y))/(1.0+dt*(u0_x(ip) + v0_y(ip)))*phi
      end do

      return
      end
*-----------------------------------------------------------------------*
*This subroutine solves the 2D Advection Equation
*using the Backward Euler with a predictor-corrector strategy
*Written by F.X. Giraldo on 2/95
*-----------------------------------------------------------------------*
      subroutine matsuno(phim,phi0,phip,um,u0,up,
     $                   vm,v0,vp,coord,intma,npoin,nelem,dt,dx,dy,
     $                   node,nx,ny,ym,nd,mxpoi,mxele,imax)
      implicit real*8(a-h,o-z)
      dimension phim(mxpoi), phi0(mxpoi), phip(mxpoi)
      dimension um(mxpoi), u0(mxpoi), up(mxpoi)
      dimension vm(mxpoi), v0(mxpoi), vp(mxpoi)
      dimension coord(mxpoi,2)
      integer intma(mxele,nd), node(imax,imax)
c
c     Loop through the points and integrate using Forward Time
c     and Centered Space...
c
```

```
*     Predictor Stage (forward Euler)

      do j=1,ny
         j1=j-1
         j2=j+1
         if (j1.lt.1) j1=ny-1
         if (j2.gt.ny) j2=2
         do i=1,nx
            i1=i-1
            i2=i+1
            if (i1.lt.1) i1=nx-1
            if (i2.gt.nx) i2=2
c
c           Set up the nodes in X and Y
c
            ip=node(i,j)
            ip1=node(i1,j)
            ip2=node(i2,j)
            jp1=node(i,j1)
            jp2=node(i,j2)
c
c           integrate PHI
c
            phim(ip)=phi0(ip)
     $     -0.5*dt/dx*u0(ip)*( phi0(ip2)-phi0(ip1) )
     $     -0.5*dt/dy*v0(ip)*( phi0(jp2)-phi0(jp1) )
c
c        integrate U
c
            um(ip)=u0(ip)
c
c        integrate V
c
            vm(ip)=v0(ip)
         end do

      end do

*     Corrector Stage (backward Euler)

      do j=1,ny
         j1=j-1
         j2=j+1
```

```
         if (j1.lt.1) j1=ny-1
         if (j2.gt.ny) j2=2
         do i=1,nx
            i1=i-1
            i2=i+1
            if (i1.lt.1) i1=nx-1
            if (i2.gt.nx) i2=2
c
c           Set up the nodes in X and Y
c
            ip=node(i,j)
            ip1=node(i1,j)
            ip2=node(i2,j)
            jp1=node(i,j1)
            jp2=node(i,j2)
c
c           integrate PHI
c
            phip(ip)=phi0(ip)
     $     -0.5*dt/dx*um(ip)*( phim(ip2)-phim(ip1) )
     $     -0.5*dt/dy*vm(ip)*( phim(jp2)-phim(jp1) )
c
c        integrate U
c
            up(ip)=u0(ip)
c
c        integrate V
c
            vp(ip)=v0(ip)
         end do

      end do
c
c     Apply the Periodic Boundary Conditions
c
      do j=1,ny
         i1=node(1,j)
         i2=node(nx,j)
         phip(i1)=phip(i2)
         up(i1)=up(i2)
         vp(i1)=vp(i2)
      end do

      do i=1,nx
```

```
         i1=node(i,1)
         i2=node(i,ny)
         phip(i1)=phip(i2)
         up(i1)=up(i2)
         vp(i1)=vp(i2)
      end do

 1000  continue

      return
      end
*---------------------------------------------------------------------*
*This subroutine writes the output.  It is currently set only to
*print the concentration (or color) function at each node point.
*Written by F.X. Giraldo on 2/95
*---------------------------------------------------------------------*
      subroutine output(phi,u,v,npoin,time,nx,ny,mxpoi)
      implicit real*8(a-h,o-z)
      dimension phi(mxpoi), u(mxpoi), v(mxpoi)

      pi=4.0*atan(1.0)
      write(1,'(2(i6,1x),e16.8)'),nx,ny,time/(2.0*pi)
      write(1,'(e12.4)')(phi(ip), ip=1,npoin)
      return
      end
*---------------------------------------------------------------------*
*These next 4 subroutines construct the Hermitian Interpolation functions
*required by the semi-Lagrangian method.
*Obtained from Numerical Recipes.
*Written by F.X. Giraldo on 2/95
*---------------------------------------------------------------------*
      subroutine splie2(df,f,coord,nx,ny,node,mxpoi,imax)
      implicit real*8(a-h,o-z)
      parameter (nmax=3000)
c
c     global arrays
c
      dimension coord(mxpoi,2), f(mxpoi), df(mxpoi)
      integer node(imax,imax)
c
c     local arrays
c
      dimension x(nmax), y(nmax), y2(nmax)
```

```
      do j=1,ny
         do i=1,nx
            y(i)=f(node(i,j))
            x(i)=coord(node(i,j),1)
         end do
         call spline(y2,y,x,nx,idum,nmax)
         do i=1,nx
            df(node(i,j))=y2(i)
         end do
      end do

      return
      end
*-----------------------------------------------------------------------*
      subroutine splin2(fd,coord,f,df,xd,yd,nx,ny,node,mxpoi,imax)
      implicit real*8(a-h,o-z)
      parameter (nmax=3000)
c
c     global arrays
c
      dimension coord(mxpoi,2), f(mxpoi), df(mxpoi)
      integer node(imax,imax)
c
c     local arrays
c
      dimension x(nmax), y(nmax), y2(nmax), y22(nmax)

      do j=1,ny
         do i=1,nx
            y(i)=f(node(i,j))
            y2(i)=df(node(i,j))
            x(i)=coord(node(i,j),1)
         end do
         call splint(fd,x,y,y2,xd,nx,nmax)
         y22(j)=fd
      end do
      do j=1,ny
         x(j)=coord(node(1,j),2)
         y(j)=y22(j)
      end do
      call spline(y2,y,x,ny,idum,nmax)
      call splint(fd,x,y,y2,yd,nx,nmax)

      return
```

```
      end
*-----------------------------------------------------------------------*
      subroutine spline(y2,y,x,n,idum,mx)
      implicit real*8(a-h,o-z)
      parameter (nmax=3000)
      dimension y(mx), y2(mx), x(mx), u(nmax)

      if (mx.gt.nmax) then
         write(*,'(" Must expand NMAX in Subroutine Spline")')
         stop
      endif

      y2(1)=0.0
      u(1)=0.0
      do i=2,n-1
         sig=(x(i)-x(i-1))/(x(i+1)-x(i-1))
         p=sig*y2(i-1) + 2.0
         y2(i)=(sig - 1.0)/p
         u(i)=(6.0*((y(i+1)-y(i))/(x(i+1)-x(i)) - (y(i)-y(i-1))
     $        /(x(i)-x(i-1)))/(x(i+1)-x(i-1))-sig*u(i-1))/p
      end do
      y2(n)=0.0
      do i=n-1,1,-1
         y2(i)=y2(i)*y2(i+1) + u(i)
      end do

      return
      end
*-----------------------------------------------------------------------*
      subroutine splint(yd,x,y,y2,xd,n,mx)
      implicit real*8(a-h,o-z)
      dimension x(mx), y(mx), y2(mx)

      i1=1
      i2=n
 10   if (i2-i1.gt.1) then
         im=(i2+i1)/2
         if (xd.gt.x(im)) then
            i1=im
         else
            i2=im
         endif
         goto 10
      endif
```

```
      x1=x(i1)
      x2=x(i2)
      dx=x2-x1
      a=(x2-xd)/dx
      b=(xd-x1)/dx
      yd=a*y(i1) + b*y(i2)
     $   + ((a**3-a)*y2(i1) + (b**3-b)*y2(i2))*(dx**2)/6.0
      return
      end
*-----------------------------------------------------------------------*
*This subroutine updates the arrays PHIM,UM,VM,ALFM, PHI0,U0,V0,ALF0
*Written by F.X. Giraldo on 2.95
*-----------------------------------------------------------------------*
      subroutine update(phim,phi0,phip,um,u0,up,vm,v0,vp,alfm,alf0,
     $                  npoin,mxpoi)
      implicit real*8(a-h,o-z)
      dimension phim(mxpoi), phi0(mxpoi), phip(mxpoi)
      dimension um(mxpoi),   u0(mxpoi),   up(mxpoi)
      dimension vm(mxpoi),   v0(mxpoi),   vp(mxpoi)
      dimension alfm(mxpoi,2), alf0(mxpoi,2)

c
c     Loop through all the nodes and update
c
      do ip=1,npoin
c
c     Update F(x-2*alpha,t-dt)=F(x-alpha,t)
c
         phim(ip)=phi0(ip)
         um(ip)=u0(ip)
         vm(ip)=v0(ip)
         alfm(ip,1)=alf0(ip,1)
         alfm(ip,2)=alf0(ip,2)
c
c     Update F(x-alpha,t)=F(x,t+dt)
c
         phi0(ip)=phip(ip)
         u0(ip)=up(ip)
         v0(ip)=vp(ip)
      end do
      return
      end
```

A Matlab Program to plot the output showing the cone at various time is given

```
clg
i=0;
j=0;
fid=fopen('matlab.out','r');
ab=fscanf(fid,'%d',1);
isets=ab(1)
while (i < isets)
ab=fscanf(fid,'%d%d%f',3);
nx=ab(1)
ny=ab(2)
hour=ab(3)
count=nx*ny;
jj=1;
while (j < ny)
[aa,count]=fscanf(fid,'%e%e%e%e%e',nx);
ab(jj:jj+count-1)=aa;
j=j+1;
jj=jj+count;
end;
u=reshape(ab,nx,ny);
v=u';
c=contour(v,10);
clabel(c);
   title(['concentration after ',num2str(hour),' revolutions'])
   print c -dps -append
i=i+1;
j=0;
end;
```

Conclusion

We have developed a bilinear finite element Fortran code to solve the two dimensional advection equation on a unix-based SUN Sparc 10 workstation. The stability of the method is analyzed. We have also developed a semi-Lagrangian finite element code. These codes were experimented with in solving the cone test problem. It is clear from the plots that the semi-Lagrangian is superior to the Eulerian finite elements, since the cone rotates back to its position without leaving a noisy trail behind.

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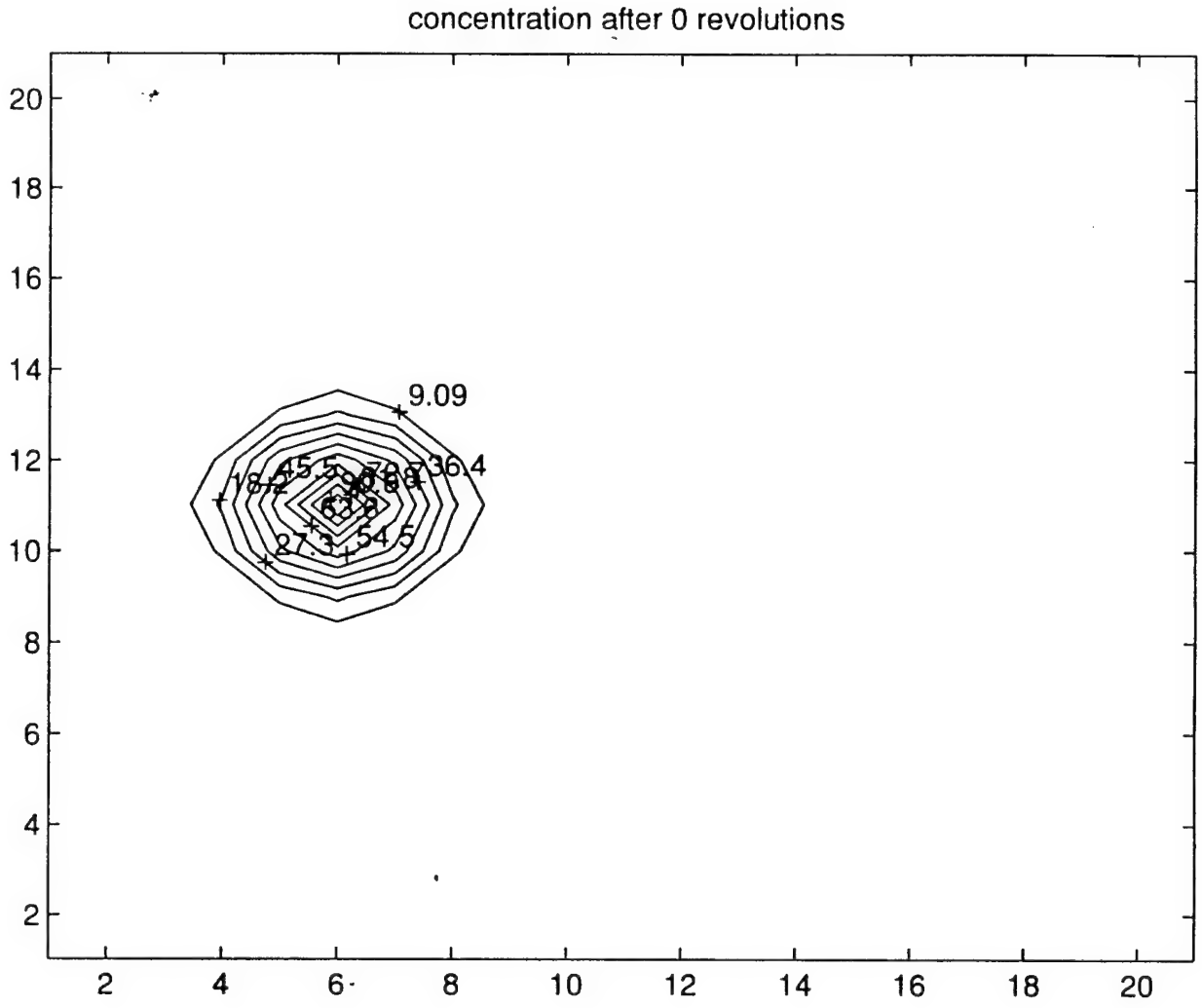
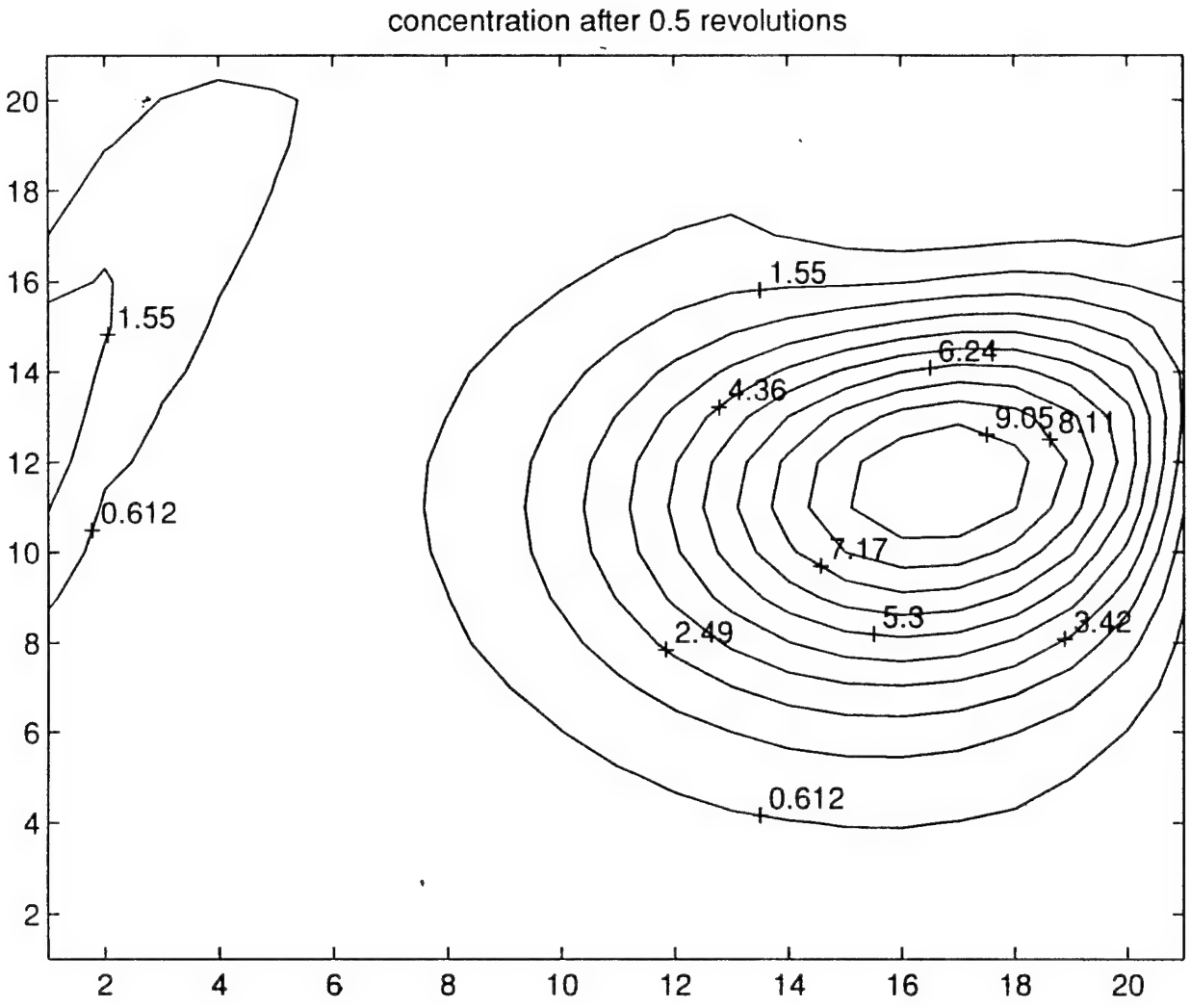
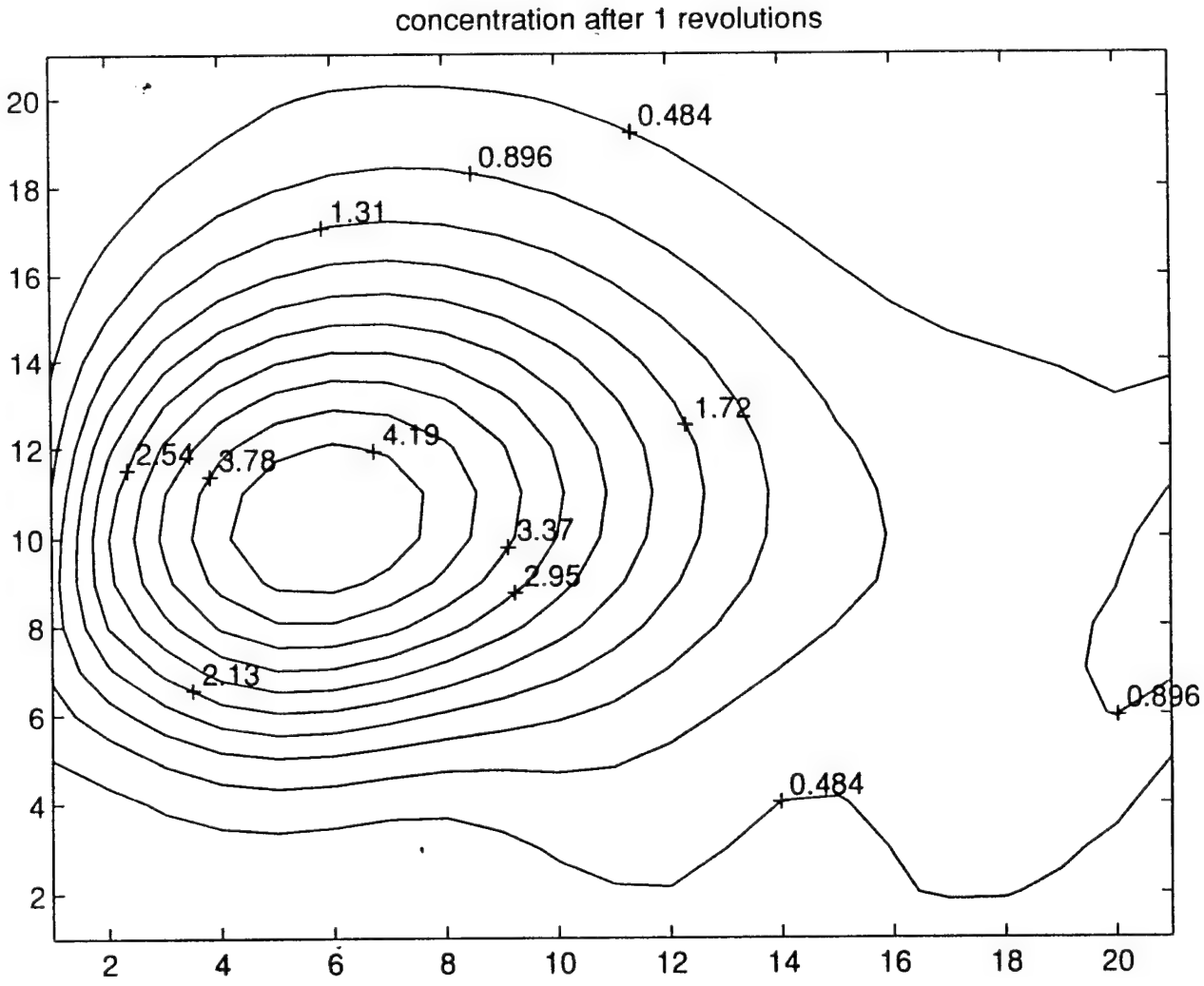
Appendix



Explicit
$n_x = n_y = 21$
$\Delta t = .0125$



Explicit
$n_x = n_y = 21$
$\Delta t = .0125$



Explicit
$n_x = n_y = 21$
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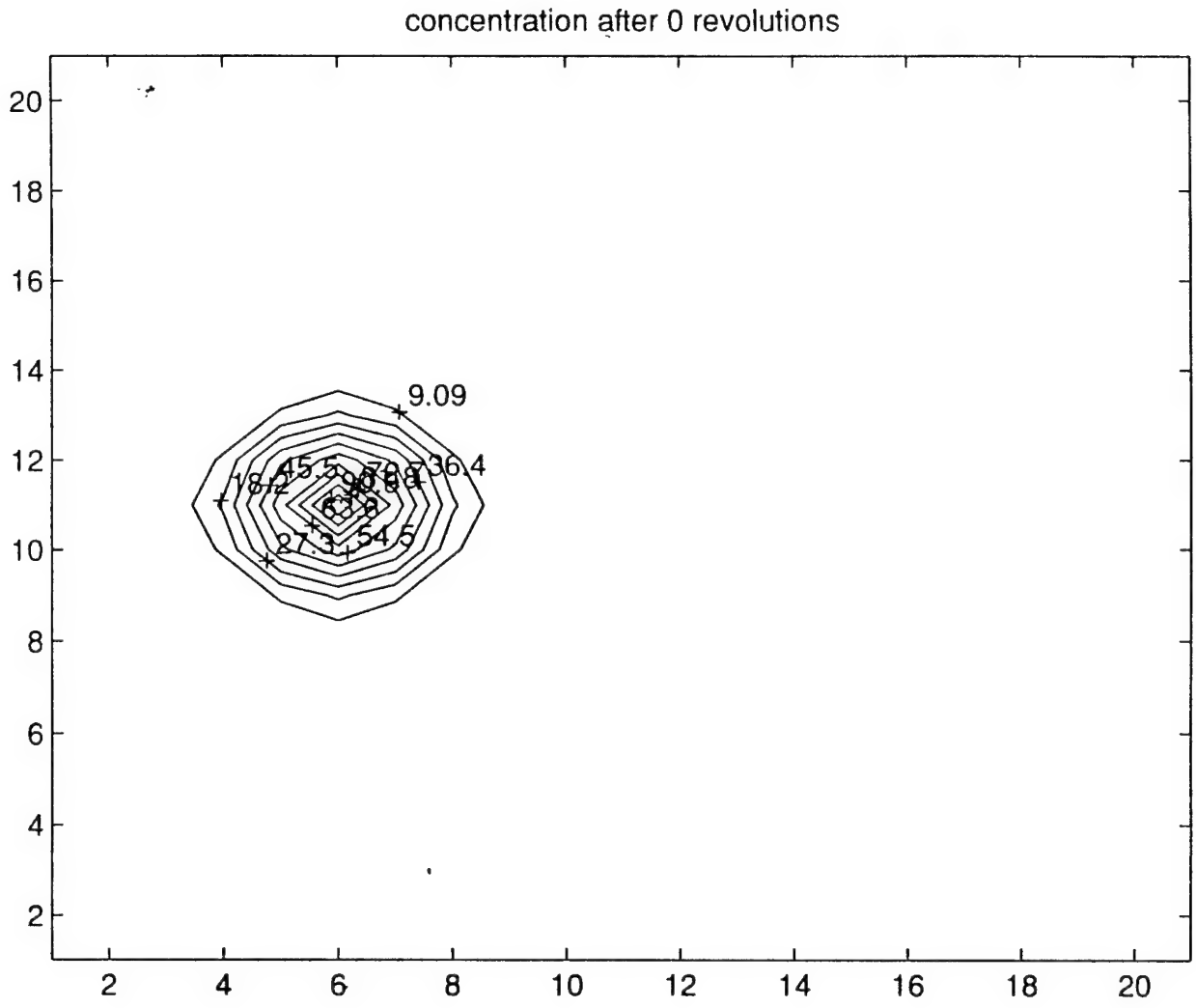
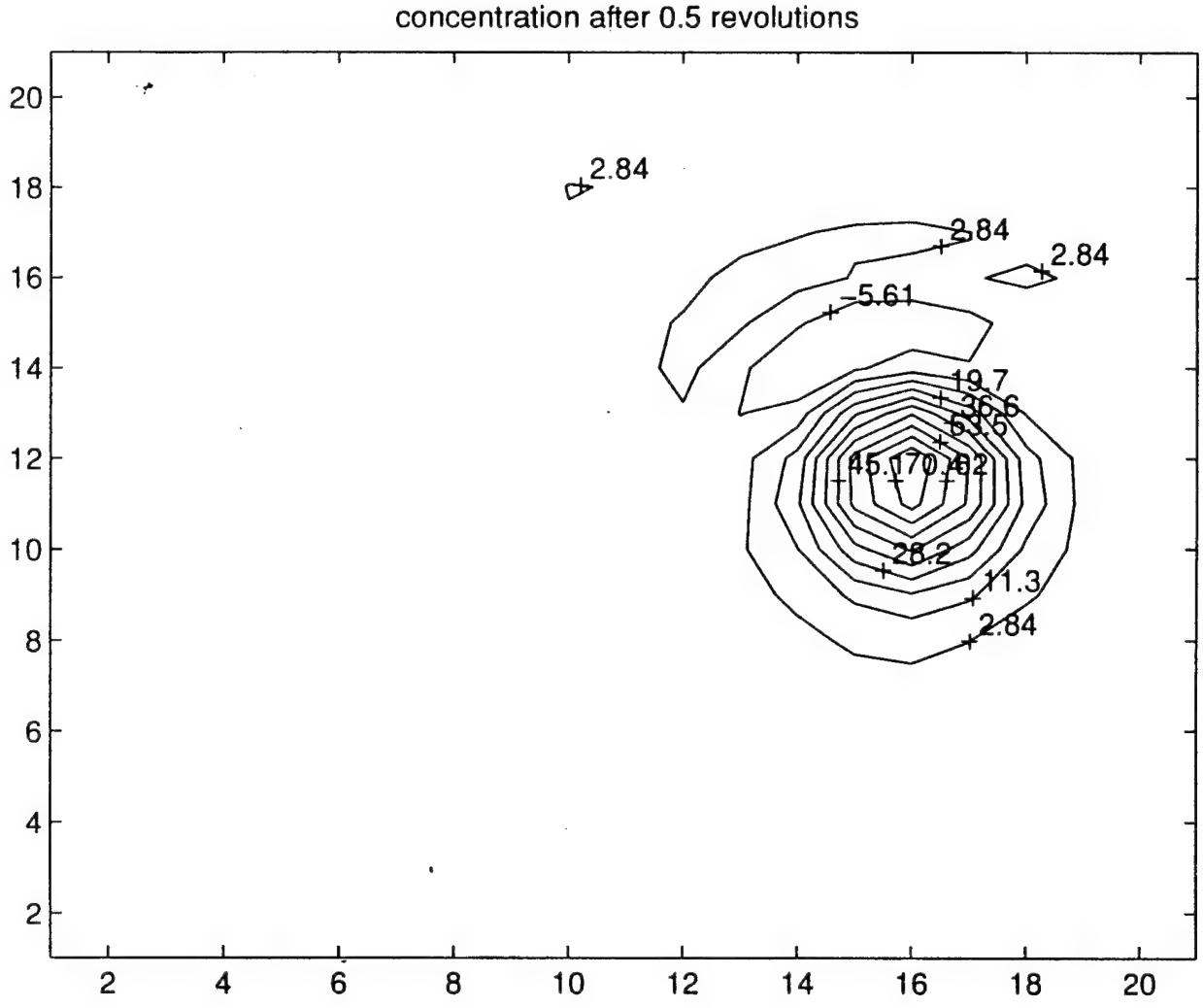
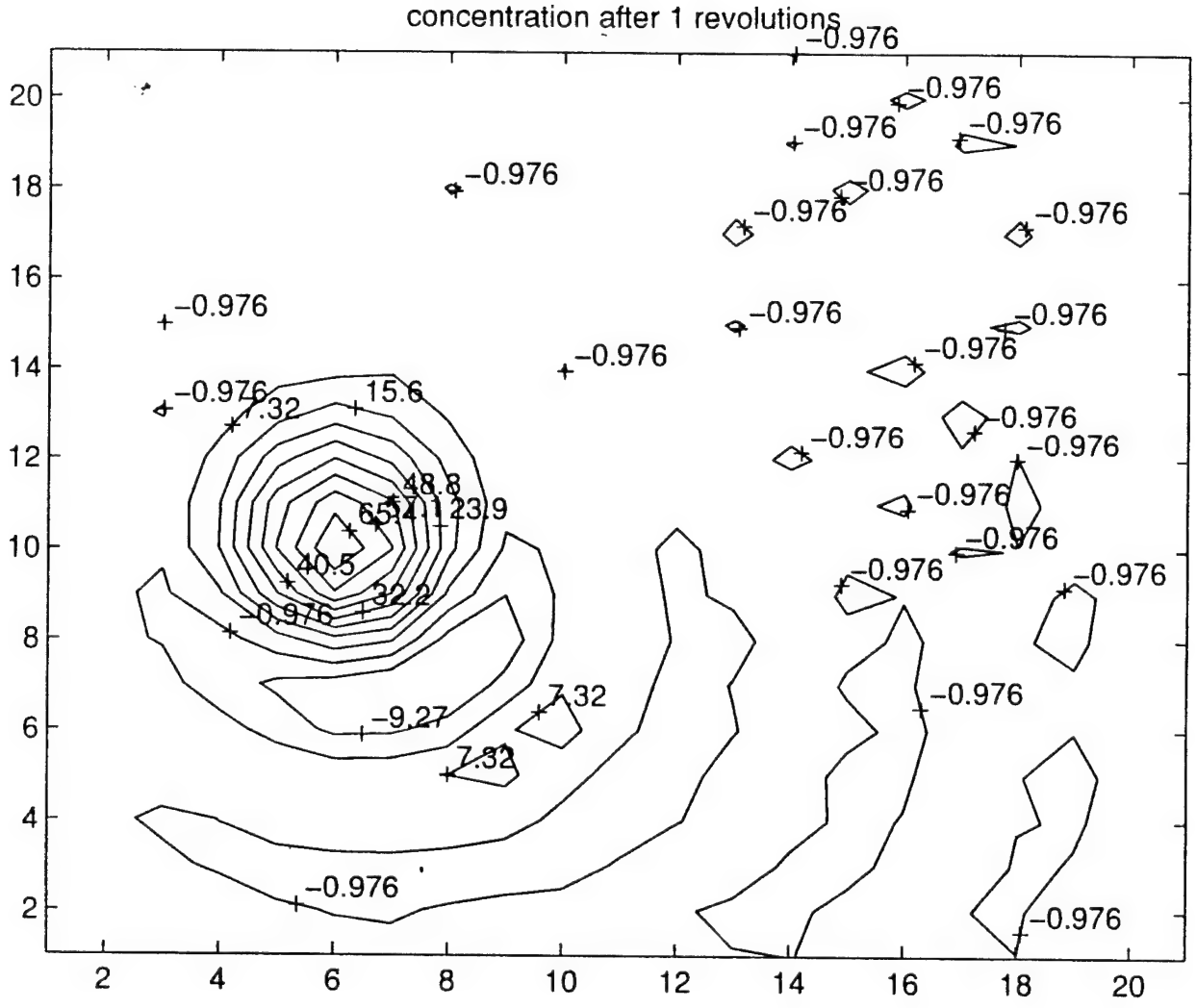


Crank-Nicholson
$n_x = n_y = 21$
$\Delta t = .0125$



Crank-Nicholson
$n_x = n_y = 21$
$\Delta t = .0125$



Crank-Nicholson
$n_x = n_y = 21$
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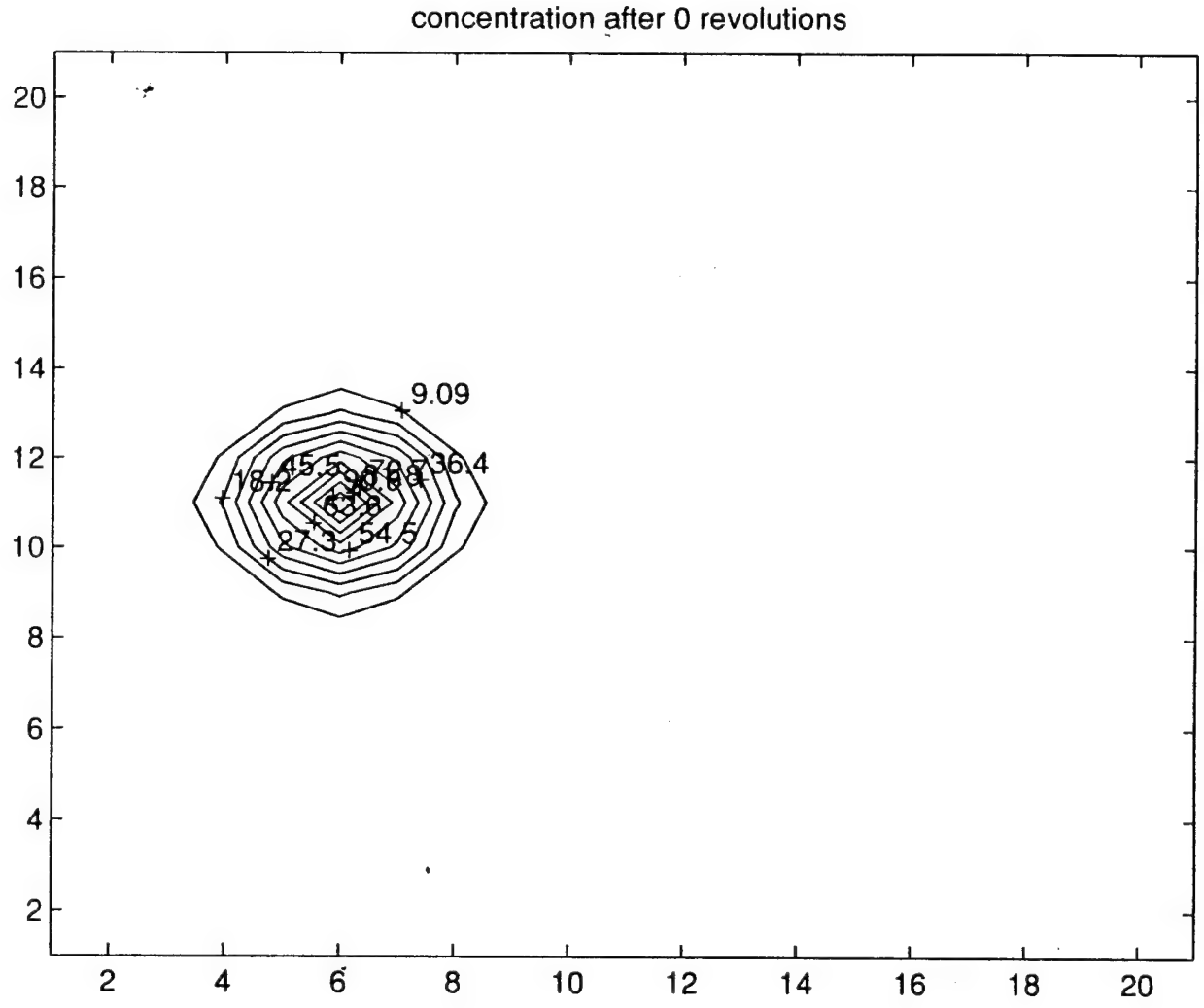

Fully implicit
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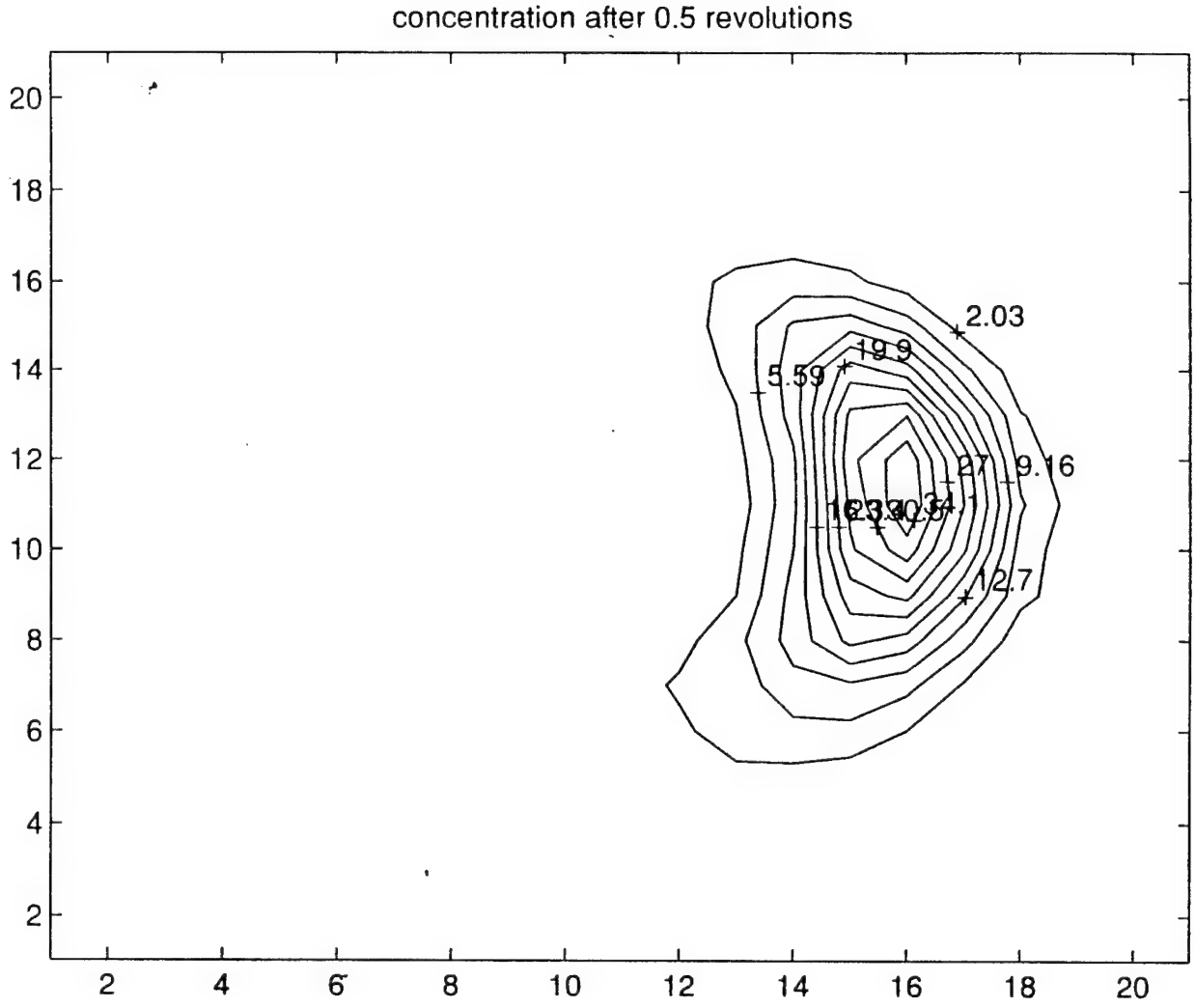
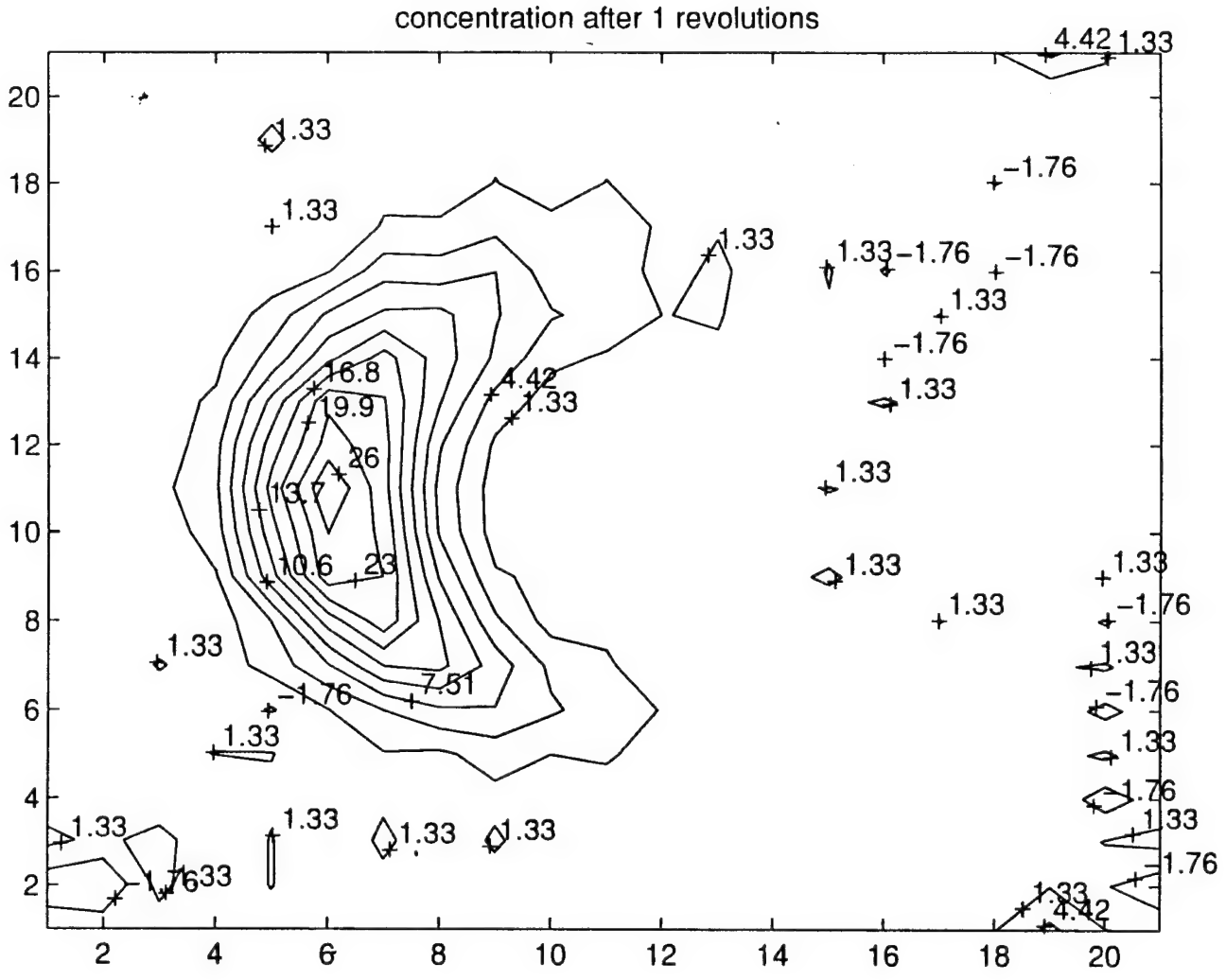
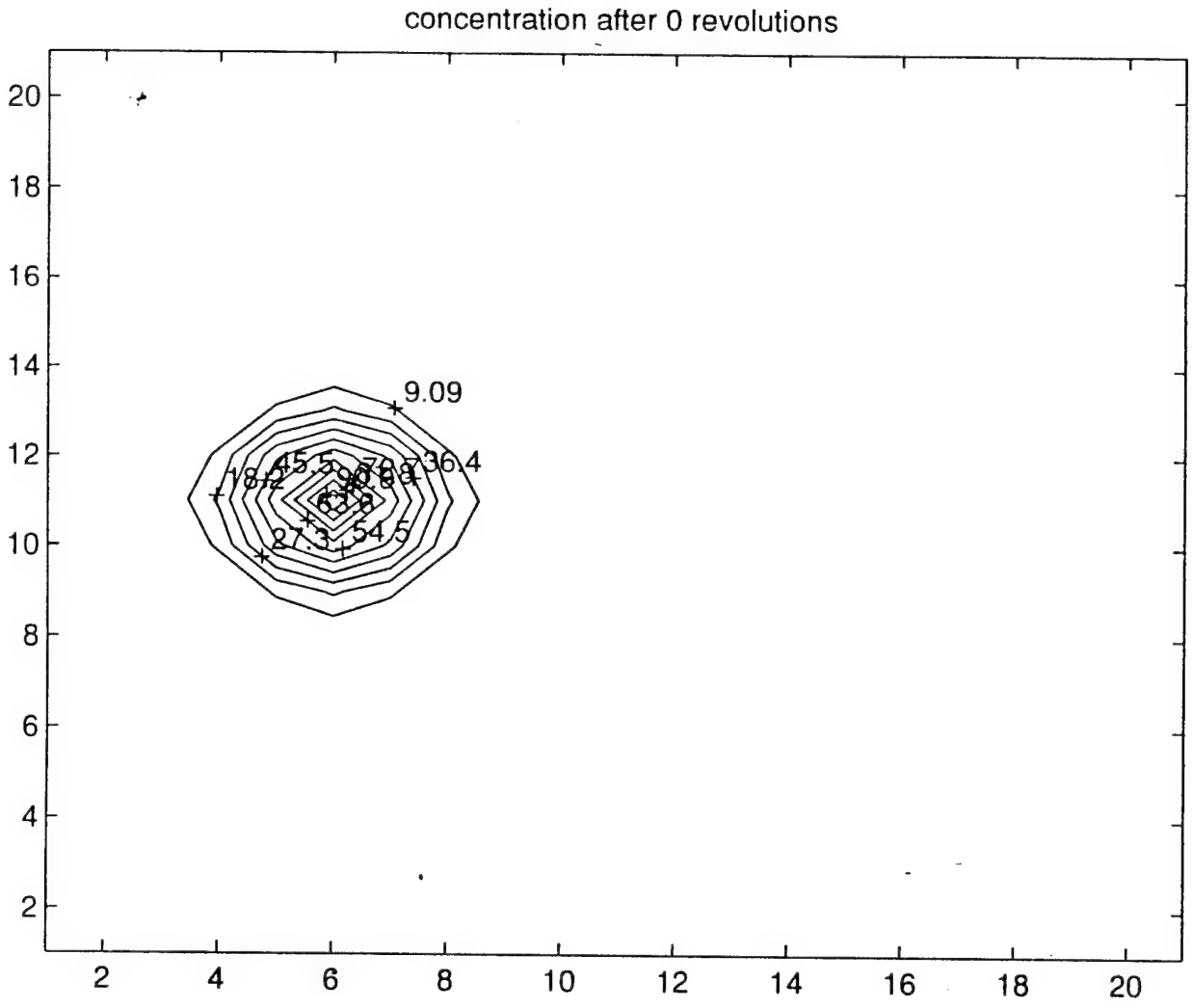
$\Delta t = .0125$



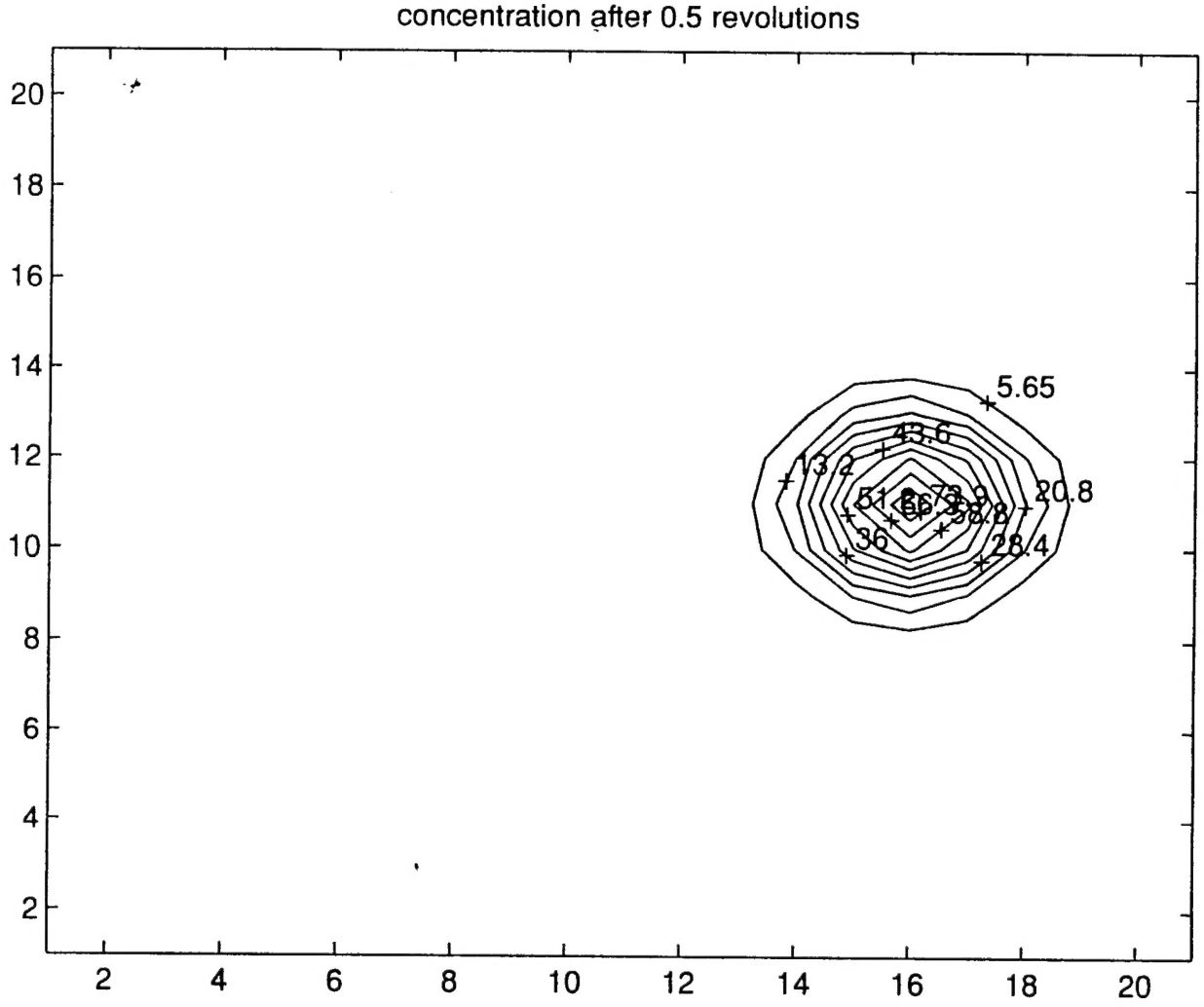
Fully implicit
$n_x = n_y = 21$
$\Delta t = .0125$



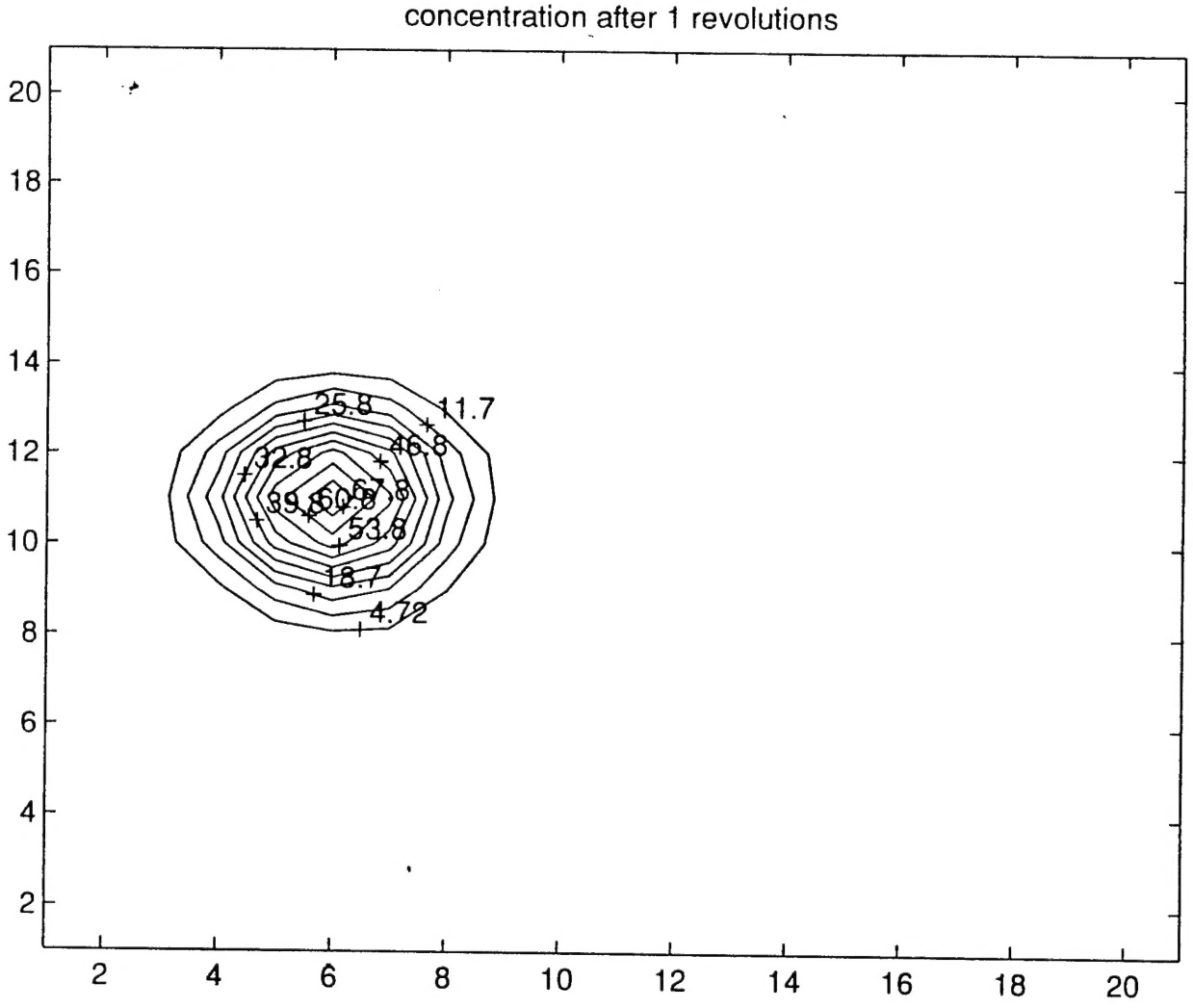
Fully implicit
$n_x = n_y = 21$
$\Delta t = .0125$



Semi-Lagrangian
$n_x = n_y = 21$
$\Delta t = .0125$



Semi-Lagrangian
$n_x = n_y = 21$
$\Delta t = .0125$



Semi-Lagrangian
$n_x = n_y = 21$
$\Delta t = .0125$

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